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D. C. 519.5

ON THE COVERINGS OF FIBRATIONS (II)

BY

FULGA POP and I. POP

The following corollary of the Theorem 1 of the first part [3] of this paper, generalizes the classification theorem for the covering spaces [2, Theorem 13, p. 109].

Corollary 1. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a Hurewicz fibration with F, E, B connected, locally path connected and semi-locally 1-connected spaces. We suppose that $i_* : \pi_1(F) \longrightarrow \pi_1(E)$ is injective and there exists a right inverse $k : \pi_1(B) \longrightarrow \pi_1(E)$ of $p_* : \pi_1(E) \longrightarrow \pi_1(B)$.*

If H_1 is a subgroup of the group $\pi_1(F)$ and H_2 is a subgroup of the group $\pi_1(B)$ such that $H = i_*(H_1) \,.\, k(H_2)$ is a subgroup of the group $\pi_1(E)$, then, there exists a connected covering fibration of p

$$\begin{tikzcd}
\tilde{F} \arrow[r, "\tilde{i}"] \arrow[d, "\pi"'] & \tilde{E} \arrow[r, "\tilde{p}"] \arrow[d, "\tilde{\pi}"'] & \tilde{B} \arrow[d, "\pi'"'] \\
F \arrow[r, "i"] & E \arrow[r, "p"] & B
\end{tikzcd}$$

such that $\pi_*(\pi_1(\tilde{F})) = H_1$, $\tilde{\pi}_*(\pi_1(\tilde{E})) = H$ and $\pi'_*(\pi_1(\tilde{B})) = H_2$.

Proof. Let $\pi : \tilde{F} \longrightarrow \tilde{F}$ and $\pi' : \tilde{B} \longrightarrow B$ be two covering maps such that $\pi^*(\pi_1(\tilde{F})) = H_1$ and $\pi'_*(\pi_1(\tilde{B})) = H_2$. We define $h = k \circ \pi'_* : \pi_1(\tilde{B}) \longrightarrow \pi_1(E)$ and we can apply the Theorem 1 of [3].

Corollary 2. *Let E, F be two connected locally path connected and semi-locally 1-connected spaces and let $F \xrightarrow{i} E \xrightarrow{p} B$ be a Hurewicz fibration.*

Suppose that B has a contractible covering space $\tilde{B}$. Then, every connected covering space $\tilde{F}$ of the space F is homotopically equivalent with a covering space of the space E.

Proof. Consider the covering maps $\pi : \tilde{F} \longrightarrow F$ and $\pi' : \tilde{B} \longrightarrow B$. We have $\pi_1(B) = \{1\}$ and $\pi_2(B) \simeq \pi_2(\tilde{B}) = \{1\}$. According to the exact homotopy sequence of the fibration p it follows that $i_* : \pi_1(F) \longrightarrow \pi_1(E)$ is injective and, since $\tilde{B}$ is a simple connected space, we

can apply the Theorem 1 of [3]. Hence, there exists a Hurewicz fibration $\widetilde{F} \xrightarrow{\widetilde{i}} \widetilde{E} \xrightarrow{\widetilde{p}} \widetilde{B}$, with $\widetilde{E}$ a covering space of the space E. Since $\widetilde{B}$ is a contractible space, it follows [2, Corollary 15, p. 137] that $\widetilde{p}$ is homotopically equivalent with the trivial fibration $\widetilde{F} \times \widetilde{B} \longrightarrow \widetilde{B}$. That is $\widetilde{E}$ is homotopically equivalent with $\widetilde{F} \times \widetilde{B}$, hence with the space $\widetilde{F}$.

Example. If $B = S^1$ then we have $\widetilde{B} = R$.

Proposition 1. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a Hurewicz fibration, let $\pi : \widetilde{F} \longrightarrow F$ be a covering map and let $f : B' \longrightarrow B$ be a continuous map.*

If p has a 1_B - covering fibration extending π, then the pullback fibration $f^* p$ has a $1_{B'}$ - covering fibration extending π.

Proof. Let $\widetilde{F} \xrightarrow{\widetilde{i}} \widetilde{E} \xrightarrow{\widetilde{p}} \widetilde{B}$ be the covering fibration of p extending π. Then, the pullback fibration $f^* \widetilde{p}$ is the required covering fibration of the fibration $f^* p$.

In order to illustrate the Theorem 1 of [3], we remember that a fibration $F \xrightarrow{i} E \xrightarrow{p} B$ is called oriented if the group $\pi_1(B)$ operates trivially on $\zeta(F)$, where $\zeta(F)$ is the group of homotopy classes of self homotopy equivalences.

Theorem 3. *Let F be a finite CW-complex which is a connected space such that $\chi(F) \neq 0$.*

Then, for every oriented Hurewicz fibration $F \xrightarrow{i} E \xrightarrow{p} B$, with B a connected CW-complex and for any two connected covering spaces $\pi : \widetilde{F} \longrightarrow F$, $\pi' : \widetilde{B} \longrightarrow B$, there exists a π'-covering fibration of p extending π.

Proof. Under the conditions of the theorem, the spaces F, E are connected, locally path connected and since they are locally contractible spaces they are also semi-locally 1-connected spaces, [4]. By a theorem of J. Stasheff [4] we can consider the universal fibration $F \xrightarrow{i} \to UE \xrightarrow{u} B_{H(F)}$.

Since $B_{H(F)}$ is a connected, locally path connected and semilocally 1-connected space, it follows [2, Corollary 15, p. 112], that a universal covering space $\widetilde{B}'$ of the space $B_{H(F)}$ is a simple connected space.

Let $\widetilde{\pi}' : \widetilde{B}' \longrightarrow B_{H(F)}$ be a universal covering map and let $F \xrightarrow{\bar{i}} D \xrightarrow{\widetilde{\pi}' \circ u} \widetilde{B}'$ be the fibration classified by the map $\widetilde{\pi}'$. We have the following exact sequence:

$$\ldots \to \pi_2(\widetilde{B}') \xrightarrow{d} \pi_1(F) \xrightarrow{\bar{i}_*} \pi_1(D) \longrightarrow \pi_1(\widetilde{B}') \to \ldots$$

But

$$\pi_2(\widetilde{B}') \simeq \pi_2(B_{H(F)}) \simeq \pi_1(H(F)),$$

[4], where $H(F)$ is the topological monoïd of the self homotopy equivalences of F. Hence, we have the following exact sequence:

$$\ldots \to \pi_1(H(F), 1_F) \xrightarrow{\bar{d}} \pi_1(F, x_0) \xrightarrow{\bar{i}_*} \pi_1(D, x_0) \to \ldots,$$

where $\bar{d}$ is defined in the following way: if $\theta : I = [0,1] \longrightarrow H(F)$ is a path satisfying the condition $\theta(0) = \theta(1) = 1_F$, that is $\theta(0)(x) = \theta(1)(x) = x$, for every point $x \in F$, we define the path $\omega : I \longrightarrow F$ by $\omega(t) = \theta(t)(x_0)$, with $x_0 \in F$ a fixed point, and then $\bar{d}([\theta]) = [\omega]$.

We have $\operatorname{Ker} \tilde{i}_* = \operatorname{Im} \tilde{d} = G_1(F)$, where $G_1(F)$ is Gottlieb's group [5]. Since F is a compact space, the condition $\chi(F) \neq 0$ implies $G_1(F) = 0$ [1]. It follows that $\tilde{i}_*$ is injective. By the Theorem 1 of [3] the fibration $\tilde{\pi}'^* u$ has a $1_{\tilde{F}}$-covering fibration extending π. We denote this covering fibration by $\tilde{F} \xrightarrow{\tilde{i}} \tilde{D} \xrightarrow{\tilde{u}} \tilde{B}'$.

If p is an oriented fibration, then this fibration is a pullback $f^*(\tilde{\pi}'^* u)$ [1] for a continuous map $f : B \longrightarrow \tilde{B}'$. By the Proposition 1, the fibration $f^* \tilde{u}$ is a 1_B-covering fibration of p extending π. Then, the pullback $\pi'^*(f^* \tilde{u})$ is a π'-covering fibration of p, extending π, in accordance with the following commutative diagram

$$\begin{tikzcd}
F \arrow[r, "i"] & E \arrow[r, "p"] & B \\
\tilde{F} \arrow[u, "\pi"] \arrow[r] & f^* \tilde{D} \arrow[u] \arrow[ur, "f^* \tilde{u}"] & \\
 & \pi'^*(f^* \tilde{D}) \arrow[u] \arrow[r, dashed, "\pi'^*(f^* \tilde{u})"] & \tilde{B} \arrow[uu, "\pi'"']
\end{tikzcd}$$

and according to the fact that the composition of covering maps of connected locally path connected spaces is a covering map [6].

Corollary 3. *If in the Theorem 3, $F \xrightarrow{i} E \xrightarrow{p} B$ is a locally trivial bundle with B a polyhedron and such that p is oriented as a Hurewicz fibration, then there exists a π'-covering of p, extending π, which is a locally trivial bundle.*

Proof. By the Theorem 3 there exists $\tilde{F} \xrightarrow{\tilde{i}} \tilde{E} \xrightarrow{\tilde{p}} \tilde{B}$ a π'-covering of p extending π. By the Theorem 2 of [3] $\tilde{p}$ is a locally trivial bundle.

Lemma 1. *Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a Hurewicz fibration. We suppose that F is a finite CW-complex which is a connected space and such that $\chi(F) \neq 0$.*

If B is a CW-complex, then $i_* : \pi_1(F) \longrightarrow \pi_1(B)$ is injective.

Proof. Let $F \xrightarrow{j} UE \xrightarrow{u} B_{H(F)}$ be the universal fibration [4]. Then, p is a pullback fibration $f^* u$, for a continuous map $f : B \longrightarrow B_{H(F)}$, and we have a commutative diagram:

$$\begin{tikzcd} F \arrow[r, "j"] \arrow[dr, "i"'] & UE \arrow[r, "u"] & B_{H(F)} \\ & E \arrow[u, "g"] \arrow[r, "p"] & B \arrow[u, "f"'] \end{tikzcd}$$

Hence $g \circ i = j$, that is $g_* \circ i_* = j_*$. But $\chi(F) \neq 0$ implies that $j_* : \pi_1(F) \to \pi_1(UE)$ is injective (Theorem 3). It follows that i_* is also injective.

Proposition 2. *Let (E, F) be a pair of CW-complexes such that F is a connected and finite CW-complex and such that $\chi(F) \neq 0$.*

We suppose that F is the fibre of a Hurewicz fibration having E as the total space and as the base space a connected CW-complex B.

By these conditions, every connected covering space of the space F can be extended to a covering space of the space E.

Proof. Let $F \xrightarrow{i} E \xrightarrow{p} B$ be a Hurewicz fibration with B a connected CW-complex. By a theorem of J. Stasheff [4], it follows that E is a CW-complex and, since F and B are path connected spaces, it follows that E is a path connected space.

Let $\pi : \widetilde{F} \to F$ be a connected covering space and let $\pi' : \widetilde{B} \to B$ be a universal covering space. Since B is connected, locally path connected and semi-locally 1-connected space, if follows [2, Corollary 15, p. 112] that B is a simple connected space.

Using Lemma 1, we can apply the Theorem 1 of [3]. It follows that there exists a π'-covering fibration of p, $\widetilde{F} \xrightarrow{\tilde{i}} \widetilde{E} \xrightarrow{\tilde{p}} \widetilde{B}$, extending π. That is, there exists a covering map $\tilde{\pi} : \widetilde{E} \to E$ such that we have the commutative diagram:

$$\begin{tikzcd} F \arrow[r, "i"] & E \\ \widetilde{F} \arrow[u, "\pi"] \arrow[r, "\tilde{i}"] & \widetilde{E} \arrow[u, "\tilde{\pi}"'] \end{tikzcd}$$

Corollary 4. *Let F be a finite and connected CW-complex such that $\chi(F) \neq 0$ and let $F \xrightarrow{j} UE \xrightarrow{u} B_{H(F)}$ be the universal fibration.*

If B is a CW-complex and $f : B \to B_{H(F)}$ is a continuous map, then we denote:

$$E_f = \{(b, x) \in B \times UE / f(b) = u(x)\}.$$

By these conditions, every connected covering space of the space F can be extended to a covering space of the space E_f.

Particularly, every connected covering space of the space F can be extended to a covering space of the total space UE of the universal fibration u.

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Polytechnic Institute, Jassy,
and
University „Al. I. Cuza", Jassy

REFERENCES

1. Becker J. C., Gottlieb D. H., *Coverings of Fibrations*. Compositio Math., **26** (*2*), 119–128 (1973).
2. Spanier E. H., *Algebraic Topology*. Mc Graw-Hill, New York, 1966.
3. Pop I., Pop F., *On the Coverings of Fibrations* (I). Bul. Inst. polit. Iaşi, **XXII (XXVI)**, *3–4*, s. I (1976).
4. Stasheff J., *A Classifications Theorem for Fibre Spaces*. Topology, **2**, 239–246 (1963).
5. Gottlieb D. H., *Evaluation Subgroups of Homotopy Groups*. Amer. J. Math, **91**, 729–755 (1969).
6. Pop I., *On some Homotopy Lifting and Extension Properties for Sequences of Continuous Map*. An. Univ. „Al. I. Cuza", Iaşi, **XIX**, *1*, 103–112 (1973).

ASUPRA ACOPERIRILOR SPAŢIILOR FIBRATE (II)

(Rezumat)

În această Notă se prezintă unele consecinţe ale teoremelor 1 şi 2 demonstrate în prima parte a acestei lucrări [3]. Astfel, se generalizează teorema de clasificare a spaţiilor de acoperire ale unui spaţiu topologic (corolarul 1), se ilustrează teorema 1 pentru cazul fibrărilor Hurewicz orientate (teorema 3) şi se găsesc condiţii suficiente ca orice spaţiu de acoperire a unui subspaţiu F al spaţiului E să poată fi extins la un spaţiu de acoperire a lui E (propoziţia 2 şi corolarul 4).

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C. Z. 512.9

OBSERVAŢII ASUPRA UNEI FAMILII DE ALGEBRE NEASOCIATIVE

DE

I. ENESCU

1. In [1] s-a cercetat familia de algebre reale de ordin trei, comutative dar neasociative, dată prin

$$e_i^2 = 0, \quad i = \overline{1, 3}, \quad e_1 e_2 = \lambda e_3, \quad e_1 e_3 = \nu e_2, \quad e_2 e_3 = \mu e_1 \tag{1}$$

cu λ, μ, ν reali, supuşi anumitor condiţii.

Ne propunem acum determinarea tipurilor neizomorfe ale familiei (1), pentru toate valorile reale ale lui λ, μ, ν.

2. Vom considera următoarele posibilităţi:

$\lambda > 0$, $\mu > 0$, $\nu > 0$. 1) $\lambda \neq \mu \neq \nu \neq \lambda$; 2) $\lambda = \mu \neq \nu$; 3) $\lambda = \mu = \nu$.

$\lambda > 0$, $\mu > 0$. 4) $\lambda \neq \mu$, $\nu = 0$; 5) $\lambda = \mu$, $\nu = 0$.

6) $\lambda = \mu = 0$, $\nu > 0$;

7) $\lambda > 0$, $\mu < 0$, $\nu = 0$; 8) $\lambda > 0$, $\mu > 0$, $\nu < 0$, $\lambda \neq \mu$;

9) $\lambda = \mu > 0$, $\nu < 0$.

Variantele ce se obţin prin schimbarea între ei a lui λ, μ, ν precum şi acelea ce provin prin schimbarea simultană a semnului celor trei parametri, conduc la tipuri izomorfe, [1], din care motiv nu au mai fost menţionate. De asemenea, cazul $\lambda = \mu = \nu = 0$ conduce la algebra nulă de ordin trei.

3. Cazurile 1), 2), 3) sînt izomorfe şi se pot reprezenta prin

$$u_i^2 = 0, \quad i = \overline{1, 3}; \quad u_i u_j = u_k, \quad i \neq j \neq k \neq i, \tag{2}$$

aşa cum se constată prin respectiv schimbările de bază

$$u_1 = \frac{\nu}{\lambda^2 \sqrt{p}} e_1, \quad u_2 = \frac{\mu}{\lambda^3 \sqrt{q}} e_2, \quad u_3 = \frac{\mu\nu}{\lambda^3 \sqrt{pq}} e_3, \quad p = \left(\frac{\nu}{\lambda}\right)^3; \quad q = \left(\frac{\mu}{\lambda}\right)^3;$$

$$u_1 = \frac{1}{\lambda \sqrt{k}} e_1, \quad u_2 = \frac{1}{\lambda} e_2, \quad u_3 = \frac{1}{\lambda \sqrt{k}} e_3; \quad k = \frac{\nu}{\lambda};$$

$$u_1 = \frac{1}{\lambda} e_1, \quad u_2 = \frac{1}{\lambda} e_2, \quad u_3 = \frac{1}{\lambda} e_3.$$

4. Cazurile 4) şi 5) sînt izomorfe şi se reprezintă prin

(3) $$u_i^2 = 0, \quad i = \overline{1, 3}; \quad u_1 u_2 = 0, \quad u_1 u_3 = u_2, \quad u_2 u_3 = u_1,$$

aşa cum se vede schimbînd baza prin respectiv:

$$u_1 = \frac{1}{\lambda\sqrt{\lambda\mu}} e_1, \quad u_2 = \frac{1}{\lambda\mu} e_2, \quad u_3 = \frac{1}{\sqrt{\lambda\mu}} e_2;$$

$$u_1 = e_1, \quad u_2 = e_3, \quad u_3 = \frac{1}{\lambda} e_2.$$

5. Cazul 6) conduce la tipul

(4) $$u_i^2 = 0; \quad i = 1, 2, 3; \quad u_1 u_2 = u_2 u_3 = 0, \quad u_1 u_3 = u_2,$$

cum arată schimbarea de bază

$$u_1 = \frac{1}{\sqrt{\nu}} e_1, \quad u_2 = e_2, \quad u_3 = \frac{1}{\sqrt{\nu}} e_3.$$

6. Cazul 7) conduce prin schimbarea de bază:

$$u_1 = \frac{1}{\lambda\sqrt{-\lambda\mu}} e_1, \quad u_2 = \frac{1}{\lambda\mu} e_3, \quad u_3 = \frac{1}{\sqrt{-\lambda\mu}}$$

la

(5) $$u_i^2 = 0, \quad i = \overline{1, 3}; \quad u_1 u_2 = 0, \quad u_1 u_3 = -u_2, \quad u_2 u_3 = u_1.$$

7. Cazurile 8) şi 9) conduc tot la un singur tip, anume:

(6) $$u_i^2 = 0, \quad i = \overline{1, 3}; \quad u_1 u_2 = u_3, \quad u_1 u_3 = -u_2, \quad u_2 u_3 = u_1,$$

fapt dovedit prin respectiv transformările:

$$u_1 = \frac{\nu}{\lambda^2\sqrt{-p}} e_1, \quad u_2 = \frac{\mu}{\lambda^2\sqrt{q}} e_2, \quad u_3 = \frac{\mu\nu}{\lambda^3\sqrt{-pq}} e_3; \quad p = \left(\frac{\nu}{\lambda}\right)^3; \quad q = \left(\frac{\mu}{\lambda}\right)^3;$$

$$u_1 = \frac{1}{\lambda\sqrt{-k}} e_1, \quad u_2 = \frac{1}{\lambda} e_2, \quad u_3 = \frac{1}{\lambda\sqrt{-k}} e_3; \quad k = \frac{\nu}{\lambda}.$$

Menţionăm că, din eroare, în [1], algebra (6) a fost reprezentată prin (2).

8. Tipurile (2) şi (5) sînt neizomorfe, [1]. Apoi, algebra (6) nu are idempotenţi, în timp ce (2) are asemenea elemente, deci (6) şi (2) nu sînt izomorfe. Algebra (6) nu este izomorfă nici cu (5), întrucît ultima are doi nilpotenţi independenţi ortogonali, iar prima nu. În algebra (4), patratul ei are ordinul unu, fapt ce atrage neizomorfismul faţă de toate tipurile în discuţie. Existenţa în (3) a doi nilpotenţi independenţi ortogonali, arată că acest tip nu este izomorf cu (6) şi nici cu (2) şi, conform observaţiei precedente, (3) nu este izomorf nici cu (4).

Deşi au unele proprietăţi comune, ca, de exemplu, aceea că $x^2 . x^2 = 0$ oricare ar fi x arbitrar în respectiv (3) şi (5), aceste algebre nu sînt izo-

morfe. Într-adevăr, în (3) suma nilpotenţilor ortogonali independenţi u_1, u_2 este invariată prin produsul cu al treilea nilpotent u_3, independent faţă de primii doi, în sensul că $u_3(u_1+u_2)=u_1+u_2$, în timp ce, în (5) nu există nilpotenţi cu o asemenea proprietate, cu indicele doi de nilpotenţă.

În concluzie, familia (1) are, pentru λ, μ, ν reali, şase tipuri neizomorfe reprezentate prin respectiv algebra nulă de ordin trei şi multiplicările (2) ... (6).

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Institutul politehnic, Iaşi,
Catedra de matematici

BIBLIOGRAFIE

1. Enescu I., *Asupra unei familii de algebre neasociative.* Bul. Inst. polit. Iaşi, XXI (XXV), *3–4*, s. I, 15–20 (1975).

REMARQUES SUR UNE FAMILLE D'ALGÈBRES NONASSOCIATIVES

(Résumé)

On détermine les types nonisomorphes de la famille (1), avec λ, μ, ν réels arbitraires.

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D. C. 517.91

PERIODIC AND ALMOST PERIODIC SOLUTIONS OF PARABOLIC EQUATIONS AND E. ROTHE'S METHOD

BY

DAN PETROVANU

0. **Introduction.** Consider the parabolic initial value problem

$$\Delta u = u_t + f(x, t, u), \tag{0.1}$$

$$u(x, 0) = u_0(x), \tag{0.2}$$

where Δ is the Laplace operator, $x = (x_1, \ldots, x_n) \in R^n$, R^n is the euclidean n-dimensional space (*n is an even number everywhere*), $f(x, t, u)$ is a function, continuous in

$$(D): \quad -\infty < x_i < \infty, \quad 0 \leqslant t \leqslant T, \quad |u| \leqslant A, \quad (i = 1, \ldots, n),$$

and satisfying in (D) the Lipschitz condition

$$|f(x, t, v) - f(x, t, w)| \leqslant L\,|v - w|. \tag{0.3}$$

A function $u(x, t)$ is said to be a (classical) solution of (0.1), (0.2) if $u_t, u_{x_i x_j}$ $(i, j = 1, \ldots, n)$ exist and are continuous functions in

$$(P): \quad -\infty < x_i < +\infty, \quad 0 < t \leqslant T \quad (i = 1, 2, \ldots, n),$$

$u(x, t)$ is continuous in

$$(\bar{P}): \quad -\infty < x_i < +\infty, \quad 0 \leqslant t \leqslant T \quad (i = 1, \ldots, n),$$

$u(x, t)$ satisfies (0.1) in (P) and (0.2) for any x. (Of course, $u_0(x)$ is a given, continuous function.)

The problem we are handling in this Note is the following one: Suppose $f(x, t, u)$ and $u_0(x)$ are periodic (almost periodic) in x. Can we conclude that the solution $u(x, t)$ of (0.1)—(0.2) is also periodic (almost periodic) with respect to x?

In the case $n = 1$, that is x is a scalar argument, C. Corduneanu ([5], see also [6]) has shown that if $u(x, t)$ is bounded and uniformly continuous, then $f(x, t, u)$ and $u_0(x)$ almost periodic in x imply $u(x, t)$ almost periodic in x. The proof in [5] is obtained, mainly by using E.

R o t h e's method [8] and an a priori estimate of the bounded solution of equation $y'' - \lambda^2 y = g(x)$, with $g(x)$ bounded on the whole line.

So what we need for extending the result to the equation (0.1) is an a priori estimate of an analogous kind for the unique bounded solution of $\Delta u - \lambda^2 u = g(x)$, for $g(x)$ bounded (and $n =$ even). Such an estimate is given in the sequel (see (1.6) below). Then, the proof of an analogous result is given in the case $n = 2j$, $(j = 1, 2, \ldots)$.

May we remark that we don't prove the existence of the solution but give only sufficient conditions for an existing solution to be periodic/almost periodic. For existence theorems of solutions in (D) see, for example, [7] and especially [10]. In this last paper, R o t h e's method is used in proving the existence of a solution to (0.1), (0.2).

1. Some Results Concerning the Equation $\Delta u - \lambda^2 u = g(x)$. We first state some lemmata (essentially known) which are necessary in the sequel. We emphasize that we take everywhere $n = 2j$, $j = 1, 2, \ldots$ that is the dimension of the space R^n is even. Then $\lambda > 0$ and the functions we are considering are all real valued.

D e f i n i t i o n. We shall say that a function $g(x)$, $g: R^n \to R^1$ is *Holder continuous in* R^n if

$$|g(y) - g(x)| \leqslant H\,[d(x, y)]^\alpha, \qquad 0 < \alpha \leqslant 1$$

on every compact set of R^n, where $d(x, y) = \left\{\sum_{i=1}^{n} (x_i - y_i)^2\right\}^{1/2}$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, and H depends only on the compact set we are considering. We shall say that $f(x, t, u)$, $f: R^n \times [0, T] \times [-A, A] \to R^1$ is Hölder continuous with respect to x, if we have

$$|f(y, t, u) - f(x, t, u)| \leqslant H_1 (d(x, y))^\alpha, \qquad 0 < \alpha \leqslant 1$$

on every compact set of R^n, where H_1 does not depend on $(t, u) \in [0, T] \times [-A, A]$ but only of the compact set we are considering. Obviously, if $\partial g / \partial x_i$ are continuous in R^n, then $g(x)$ is Hölder continuous in R^n etc.

L e m m a 1. *Suppose $v(x)$ is a classical solution of*

$$\Delta v - \lambda^2 v = 0, \qquad (n = 2j), \tag{1.1}$$

bounded in the whole R^n. Then $v(x) \equiv 0$.

L e m m a 2. *Suppose $g(x)$, $g: R^n \to R^1$ is bounded and Hölder continuous in R^n, $(n = 2j)$. Then the unique bounded in R^n (classical) solution of*

$$\Delta u - \lambda^2 u = g(x) \tag{1.2}$$

is

$$U_g(x) = \frac{-\lambda^{(n-2)/2}}{(2\pi)^{n/2}} \int_{R^n} K_{\frac{n-2}{2}}(\lambda r)\, r^{(2-n)/2} g(\xi)\, d\xi. \tag{1.3}$$

Furthermore, we have

$$U_1(x) = \frac{-\lambda^{(n-2)/2}}{(2\pi)^{n/2}} \int_{R^n} K_{\frac{n-2}{2}}(\lambda r)\, r^{(2-n)/2}\, d\xi \equiv -\frac{1}{\lambda^2}\,. \tag{1.4}$$

Thus, denoting for $w(x)$, $w : R^n \to R^1$ *continuous and bounded in* R^n

$$\| w \| \overset{\text{def}}{=} \sup | w(x) | \quad \textit{for} \quad x \in R^n, \tag{1.5}$$

we have

$$\| U_g \| \leqslant \frac{\| g \|}{\lambda^2}\,, \tag{1.6}$$

and $U_g(x)$ *is also Hölder continuous in* R^n, *with the same exponent* α *as the one of* $g(x)$.

L e m m a 3. *Under the assumptions of Lemma 2, if* $g(x)$ *is* ω*-periodic in* x, *then* $U_g(x)$ *is also* ω*-periodic in* x; *if* $g(x)$ *is almost-periodic in* x, *then* $U_g(x)$ *is also almost-periodic in* x.

Here $r = \left\{ \sum_{i=1}^{n} (x_i - \xi_i)^2 \right\}^{1/2}$ and $K_{(n-2)/2}(z) = K_{j-1}(z)$ is the well-known Bessel function (see [11], pp. 78—80).

Lemma 1 follows, for example, by applying Green's formula to the pair of functions $v(x)$, $K_{(n-2)/2}(\lambda r)\, r^{(2-n)/2}$ and by taking into account that

$$G(r) = \frac{-\lambda^{(n-2)/2}}{(2\pi)^{n/2}} K_{\frac{n-2}{2}}(\lambda r)\, r^{(2-n)/2} = \gamma_n K_{\frac{n-2}{2}}(\lambda r)\, r^{(2-n)/2}$$

is a fundamental solution of (1.1) (see also [1], [9], [11] p. 97). By usual means of potential theory we can easily prove that $U_g(x)$ is a bounded solution of (1.1) if $g(x)$ is bounded and Hölder continuous, whilst the uniqueness of $U_g(x)$ is a corollary to Lemma 1. (Let us mention for the interested reader, that the boundedness of solutions of more general than (1.1) or (1.2) equations (like $\Delta u - c(x)\, u = f(x)$, $c(x) \geqslant 0$) was investigated (especially for finite regions) by some authors, starting with the papers of M. B r e l o t ([2], [3], [4]).

Lemma 3 is obvious if we take into account that $U_g(x)$ is the convolution (with respect to ξ) of $G(r)$ and $g(\xi)$.

(For topics related to the use of the fundamental solution above, see also [9], [1]).

2. Periodic (almost periodic) Solutions of the Parabolic Initial Value Problem. Consider again the initial value problem (0.1)—(0.2) and suppose $f(x, t, u)$ is bounded and continuous in (D), satisfies in (D) the Lipschitz condition (0.3), and is Hölder continuous with respect to the spatial variable x. Remark that $f(x, t, u)$ can be prolonged for any real u, such as to remain Lipschitzian in u, Hölder continuous in x and continuous in (x, t, u).

Let $u(x, t)$ be the solution of problem (0.1), (0.2) and consider, as it is usual in R o t h e's method ([8], [5], [6]) the „approximate system",

$$(2.1)\quad \Delta u_s(x) = h^{-1}[u_s(x) - u_{s-1}(x)] + f(x, t_s, u_{s-1}(x)), \quad (s=1,\dots, m)$$

where $mh = T$ and $t_s = sh$. If $u_0(x)$ is supposed to be bounded and Hölder continuous with exponent α, then by Lemma 2 each of the $u_1(x),\dots,u_m(x)$ is uniquely determined as a bounded and Hölder continuous function with the same exponent α. Denote now

$$(2.2)\qquad \varepsilon_s(x) \overset{\text{def}}{=} u(x, t_s) - u_s(x), \quad (s = 1,\dots, m),$$

and $\varepsilon_0(x) = 0$. We then obtain

$$(2.3)\qquad \Delta\varepsilon_s(x) - h^{-1}\varepsilon_s(x) = -h^{-1}\varepsilon_{s-1}(x) + r_s(x),$$

$(s = 1,\dots, m)$, where

$$r_s(x) = u_t(x, t_s) - h^{-1}[u(x, t_s) - u(x, t_{s-1})] + \\ + f(x, t_s, u(x, t_s)) - f(x, t_s, u_{s-1}(x)),$$

$(s = 1,\dots, m)$. We can state now

L e m m a 4. *Suppose $f(x, t, u)$ is bounded and continuous in (D), Hölder continuous with respect to x in R^n, and satisfies (0.3) in (D). Let $u(x, t)$ be a (classical) solution of (0.1) — (0.2) and suppose $u_0(x)$ is Hölder continuous in R^n and bounded in R^n, $u(x, t)$ and $u_t(x, t)$ are uniformly continuous, whilst $u_t(x, t)$ is Holder continuous in x. Then the process described by (2.1) is convergent. Moreover, if $\Omega(h)$ is the modulus of continuity of the functions $u(x, t)$ and $u_t(x, t)$, then there exists a constant M such that*

$$(2.4)\qquad \|\varepsilon_s\| \leqslant M\Omega(h), \quad (s = 1,\dots, m).$$

P r o o f. Under our assumptions, for any fixed s, the right hand side of (2.3) is bounded and Hölder continuous in x. Thus we can apply consecutively Lemma 2 for $s = 1, 2, \dots, m$. By (1.6) we obtain

$$(2.5)\qquad \|\varepsilon_s\| \leqslant \|\varepsilon_{s-1}\| + h\|r_s\|, \quad (s = 1, \dots, m).$$

Now we have, as in [5],

$$r_s(x) = u_t(x, t_s) - u_t(x, 0_s) + f(x, t_s, u(x, t_s)) - f(x, t_s, u_{s-1}(x))$$

and we obtain from (2.5)

$$(2.6)\qquad \|\varepsilon_s\| \leqslant (1 + hL)\|\varepsilon_{s-1}\| + (1 + L)h\Omega(h),$$

$(s = 1, \dots, m)$ and thus

$$(2.7)\qquad \|\varepsilon_s\| \leqslant \Omega(h)(1 + L)L^{-1}[(1 + hL)^m - 1].$$

As $mh = T$, we can obviously take $M = (1 + L)L^{-1}[e^{LT} - 1]$.

T h e o r e m. *Let $u(x, t)$ be a (classical) solution of (0.1) — (0.2), $u: R^n \times [0, T] \to R^1$, ($n$ is an even number). Suppose $u(x, t)$, $u_0(x)$, $f(x, t, u)$ satisfy the hypotheses of Lemma 4. Then* A) *$f(x, t, u)$ is almost periodic in x and $u_0(x)$ is almost periodic in x imply $u(x, t)$ is almost periodic in x also.* B). *$f(x, t, u)$ and $u_0(x)$ are ω-periodic in x imply $u(x, t)$ is ω-periodic in x also.*

Proof A). Like in [5] take for an arbitrary $\varepsilon > 0$, h sufficiently small to have

$$4(M+1)\Omega(h) < \varepsilon \tag{2.8}$$

where M is the constant of (2.4). By applying consecutively (like in [5]) Lemma 3 to the system (2.1) one finds that there exists a number $p(\varepsilon)$ such that any n-dimensional cube of side $p(\varepsilon)$ contains an $\varepsilon/2$-almost period common to all $u_s(x)$, $(s = 1, \dots, m)$.

For any $t \in [0, T]$ there exists $t_s = sh$ such that $|t - t_s| < h$. If η is an $\varepsilon/2$-almost period common to $u_s(x)$, $(s = 0, 1, \dots, m)$, then

$$\begin{aligned} |u(x+\eta, t) - u(x, t) &\leqslant |u(x+\eta, t) - u(x+\eta, t_s)| + |u(x+\eta, t_s) - \\ &- |u_s(x+\eta)| + |u_s(x+\eta) - u_s(x)| + |u_s(x) - u(x, t_s)| + |u(x, t_s) - \\ &- u(x, t)| < 2(M+1)\,\Omega(h) + |u_s(x+\eta) - u_s(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

B). If now $f(x, t, u)$ and $u_0(x)$ are ω - periodic, then by (2.1) and Lemma 3 all the $u_s(x)$, $(s = 1, \dots, m)$ are ω - periodic: $u_s(x+\omega) = u_s(x)$ for any $x \in R^n$. If h is taken such as to have

$$2(M+1)\,\Omega(h) < \varepsilon$$

and for every $t \in [0, T]$, $t_s = sh$ is taken such as to have $|t - t_s| < h$, we obtain

$$\begin{aligned} |u(x+\omega, t) - u(x, t)| &\leqslant |u(x+\omega, t) - u(x+\omega, t_s)| + |u(x+\omega, t_s) - u(x+\omega)| + \\ &+ |u_s(x) - u(x, t_s)| + |u(x, t_s) - u(x, t)| < 2(M+1)\,\Omega(h) < \varepsilon \end{aligned}$$

for any $x \in R^n$. As ε is arbitrary, and ω independent on ε, we obtain $u(x+\omega, t) = u(x, t)$.

Remark 1. If we take instead of (0.1) the equation

$$\sum_i \sum_j a_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j} = u_t + f(x, t, u),$$

where a_{ij} are constant, $A = [a_{ij}]$ is positive definite, then all we have obtained before holds with obvious modifications. For example, instead of $G(r)$ we have to consider

$$\Gamma(R) = \beta_n K_{\frac{n-2}{2}}(\lambda R)\, R^{(2-n)/2}, \qquad R = \left\{ \sum_i \sum_j a_{ij}(x_i - \xi_i)(x_j - \xi_j) \right\}^{1/2},$$

and β^n is a convenient constant. Then lemmata 1, 2, 3 hold with obvious, slight modifications etc..

Remark 2. Furthermore, the method used above works for an equation of the form $Lu = u_t + f(x, t, u)$, with L a linear differential operator, whenever it is possible to prove for the associated equation $Lu - \lambda^2 u = g$ results analogous to lemmata 1, 2, 3.

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REFERENCES

1. Amerio L., *Sull'integrazione dell'equazione $\Delta_2 u - \lambda^2 u = f$ in un dominio di connessione qualsiasi.* Rend. Ist. Lombardo, 78 (IX), 79–102 (1944–1945).
2. Brelot M., *Étude de l'équation de la chaleur $u = c(M)u(M)$, $c(M) \geqslant 0$ au voisinage d'un point singulier du coefficient.* Ann. Sci. de l'École Norm. Sup., 48, 153–246 (1931).

3. Brelot M., *Quelques propriétés générales des intégrales bornées de $\Delta u = cu$ sur un domaine borné ouvert où c est continu.* Bull. Sc. Math., **56**, 105–117 (1932).
4. Brelot M., *Sur l'allure à la frontière de la solution du problème de Dirichlet généralisé relatif à l'équation $\Delta u = cu + f$, $c \geqslant 0$, $|f|$ borné.* Rend. Ist. Lombardo, **65**, 119–128 (1932).
5. Corduneanu C., *Solutions presque-périodiques de certaines équations paraboliques.* Mathematica Cluj, **9** (**32**), *2*, 241–244 (1967).
6. Corduneanu C., *Approximation des solutions d'une équation parabolique dans un domaine non borné.* Mathematica Cluj, **3** (**26**), *2*, 217–224 (1961).
7. Олейник О. А., Вентцель Т. Д., *Первая краевая задача и задача Коши для квазилинейных уравнений параболического типа.* Матем. Сб., **41** (**83**), 105—128 (1957).
8. Rothe E., *Zweidimensionale parabolische Randwertaufgaben als Grenzfall eindimensionaler Randwertaufgaben.* Math. Ann., **102**, 650–670 (1930).
9. Schwartz L., *Théorie des Distributions*, Tome II, Actualit. Sci. et Ind., No. *1122*, 1951.
10. Вентцель Т. Д., *Первая краевая задача и задача Коши для квазилинейного параболического уравнения со многими пространственными переменными.* Матем. Сб., **41** (**83**), 499—520 (1957).
11. Watson G. N., *Theory of Bessel Functions.* Cambridge University Press, 1922.

SOLUŢII PERIODICE ŞI APROAPE PERIODICE ALE ECUAŢIILOR PARABOLICE ŞI METODA LUI ROTHE

(Rezumat)

Prin metoda lui Rothe şi alte consideraţii se dau criterii de periodicitate în x sau aproape periodicitate în x pentru soluţiile (presupuse că există) ale problemei (0.1), (0.2).

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SUR QUELQUES RÉSULTATS DE R. MATHAR

PAR

FROIM MARCUS

À la mémoire du prof. ANTON ŞESAN

1. Dans la Note „Sur une classe particulière de congruences", M. R. Mathar [1] s'est proposé d'étudier les congruences telles *que les tangentes asymptotiques à une nappe focale et les tangentes asymptotiques à l'autre nappe focale, forment un quatern harmonique. Il nomme ces congruences, congruences* W^*.

Le but de cette Note est de montrer que les congruences W^* *sont les congruences de Waelsh.*

De cette dernière classe de congruences, nous nous sommes occupé dans diverses Notes. (Voir le troisième volume de l'excellent traité de géométrie projective differentielle de G. Bol [4]).

En effet, soit (G) une congruence de droites engendrée par une droite (g) dont y et z sont les points focaux et que nos supposons, avec R. Mathar, rapportée à ses développables u et v. On peut, avec une normalisation des coordonnées, réduire le système complètement intégrable satisfait par les coordonnées des points y et z à la forme suivante [3] :

$$y^{10} = mz\,, \quad z^{01} = ny\,,$$

$$(1.1) \quad y^{02} = ay + bz + cy^{01} + dz^{10}, \quad z^{20} = a_1 y + b_1 z + c_1^{01} + d_1 z^{10}, \quad mn \neq 0,$$

où avec la notation f^{ij} nous notons les dérivées partielles de la fonction $f = f(u, v)$ i fois par rapport à u et j fois par rapport à v.

De (1.1), on tire

$$y^{11} - (\log m)^{01} y^{10} - mny = 0\,, \quad z^{11} - (\log n)^{10} z^{01} - nmz = 0,$$

$$(1.2) \qquad y^{20} - \frac{m}{d} y^{02} + \left(\frac{b}{d} - (\log m)^{10}\right) y^{10} + \frac{mc}{d} y^{01} + \frac{am}{d} y = 0,$$

$$z^{20} - \frac{c_1}{n} z^{02} - d_1 z^{10} - \left(\frac{a_1}{n} - \frac{c_1}{n^2} n^{01}\right) z^{01} - b_1 z = 0.$$

Les invariants ponctuels et tangentiels des équations de Laplace qui correspondent aux nappes focales (y) et (z) sont respectivement :

$$h = mn - (\log m)^{11}\,, \quad k = mn,$$

$$\bar{h} = mn - c^{10}, \quad \bar{k} = mn + \left(\frac{b}{d}\right)^{01} \tag{1.3}$$

et

$$h_{-1} = mn, \quad k_{-1} = mn - (\log n)^{11},$$

$$\bar{h}_{-1} = mn + c + \frac{c_1^{01}}{c^1} + 2(\log n)^{11}, \quad \bar{k}_{-1} = mn - d_1^{01}. \tag{1.4}$$

Des conditions d'intégrabilité du système (1.1) que nous omettons, on a aussi les deux conditions suivantes :

$$\frac{c_1^{01}}{c_1} = -c - \frac{a_1}{c_1}, \quad d_1^{01} = mn - c_1 d. \tag{1.5}$$

Nous pouvons donc écrire les deux dernières conditions (1.4) sous la forme suivante :

$$\bar{h}_{-1} = mn + 2(\log n)^{11} - \frac{a^1}{c_1}, \quad \bar{k}_{-1} = c_1 d. \tag{1.6}$$

De (1.4) et (1.6), il résulte :

$$h_{-1} + k_{-1} = mn + c_1 d. \tag{A}$$

Mais si l'on suppose que les tangentes asymptotiques des deux nappes focales (y) et (z) forment un quatern harmonique, alors

$$mn + c_1 d = 0. \tag{1.7}$$

Par conséquent :

$$h_{-1} + \bar{k}_{-1} = 0, \tag{A'}$$

et d'après (5) *et* [6] *celà signifie que la congruence* (G) *est une congruence de Waelsh.* Réciproquement, *si la congruence engendrée par la droite* (g) *est une congruence de Waelsh, alors les tangentes asymptotiques des deux nappes focales* (y) *et* (z) *forment un quaterne harmonique.*

Il est utile de rappeler qu'une congruence de droites de Waelsh, est caracterisée *par la propriété qu'aux lignes asymptotiques d'une nappe focale, correspond un système de lignes conjugées sur l'autre nappe focale,* (voire [4]). Et il est immédiat que dans ce cas les *tangentes asymptotiques des deux nappes focales forment un quaterne harmonique.*

Donc les congruences W^* *de R. Mathar sont des congruences de Waelsh.*

2. Dans [1] on donne aussi les propriétés suivantes des congruences W^*.

1. Une congrunece W^* n'est projectivement applicable que sur une congruence W^*.

2. Les surfaces focales d'une congruence W^* ne peuvent être simultanément des quadriques. Nous voulons faire les observations suivantes :

La première propriété découle des conditions de Fubini [2] (tome II, page 597), *pour que deux congruences de droites à nappes focales données soient projectivement applicables.*

La deuxieme propriété est une conséquence du résultat suivant: *Les congruences à nappes focales de deux quadriques, appartiennent à des complexes linéaires. Elles sont donc des congruences W.*

Enfin nous voulons encore rappeler les résultats suivants (voir [7]).

a. *Un réseau isotherme-conjugué tel que les tangentes aux courbes du réseau engendrent des congruences de Waelsh, s'appelle réseau de Waelsh.*

b. *Les surfaces de Jonas* [2] *sont les seules surfaces sur lesquelles il pourrait exister un réseau de Waelsh.*

Théorème 1. *Il existe des surfaces Jonas non réglées qui possèdent un réseau Waelsh, mais le réseau u, v, est de type Koenigs. Ces surfaces dépendent de deux fonctions arbitraires d'un argument et d'une constante arbitraire.*

Théorème 2. *Il existe des surfaces Jonas réglées qui possèdent un réseau Waelsh, qui n'est pas de type Koenigs.*

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BIBLIOGRAPHIE

1. Mathar R., *Sur une classe particulière de congruences.* Bull. des Sciences de Liège, 43e année, *11–12*, 580–582 (1974).
2. Fubini G., Čech E., *Geometria proiettiva differenziale.* N. Zanichelli, Bologna.
3. Wilczynski E. J., *Sur la théorie générale des congruences.* Mém. de l'Académie Roy. de Belgique, 2e série, III, 1910–12, 86 pages.
4. Bol G., *Projektive Differential Geometrie.* Vandenhoeck et Ruprecht, Göttingen, 1967.
5. Marcus F., *Sur une propriété qui caractérise les surfaces de Jonas.* Mém. de l' Académie Roy. de Belgique, 5e série, 36, 906–910 (1950).
6. Marcus F., *Sur les invariants de la théorie projective différentielle des congruences de droites.* Gazeta Mat. Fizică, 409–416 (1955).
7. Marcus F., *Les réseaux Waelsh et les surfaces de Jonas*, Bul. Inst. polit. Iaşi, IX (XIII), *3–4* (1963).

ASUPRA UNOR REZULTATE ALE LUI R. MATHAR

(Rezumat)

Se arată că congruenţele de drepte studiate de R. Mathar în [1], pe care le-a denumit congruenţe W^*, sînt congruenţe Waelsh.

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CONEXIUNI PE SPAŢII RIEMANN PARTICULARE

DE

ELENA VAMANU

1. Introducere

1.1. Grupul spin (4). Fie V_4 varietatea spaţiu-timp din relativitatea generală, $g_{\alpha\beta}$ (α, $\beta, \gamma, = 0, 1, 2, 3$), tensorul metric al varietăţii şi $p_\alpha = (p^a_{\alpha b})$, $(a, b, c = 1, 2, 3, 4)$, un sistem de matrice complexe de ordin 4.

D e f i n i ţ i e. Se spune că matricele p_α formează un sistem de matrice Dirac dacă verifică relaţia :

$$p_\alpha p_\beta + p_\beta p_\alpha = -2g_{\alpha\beta}\, e\,, \tag{1}$$

e — matricea unitate de ordin 4.

Printre proprietăţile matricelor Dirac p_α amintim : a) orice matrice $A_{4\times 4}$ care comută cu p_α este de forma ce, cu c — număr complex ; b) dacă $p_{\alpha'}$ este un alt sistem de matrice Dirac, atunci există o matrice regulată λ (definită pînă la un factor complex) astfel încît

$$p_{\alpha'} = \lambda p_\alpha \lambda^{-1} \tag{2}$$

şi c) $\operatorname{Det} p_{\alpha'} = 1$.

Mulţimea matricelor complexe de ordin 4 formează un spaţiu vectorial K peste R, de dimensiune 16. O bază în acest spaţiu este dată de matricele: $e, p_\alpha p_\alpha p_\beta, p_\alpha p_\beta p_\gamma, p_\alpha p_\beta p_\gamma p_\delta$.

Spaţiul vectorial K are structură de algebră asociativă ; aceasta este algebra Clifford generată de matricele p_α, pe care o vom nota tot prin K [5].

Elementele inversabile ale lui K definesc un grup multiplicativ K^* care are structură de grup Lie. Fie M subspaţiul vectorial al spaţiului K care admite ca bază pe p_α, şi G subgrupul închis al lui K^* format din matricele λ ce verifică condiţiile :

1°. $\lambda M \lambda^{-1} CM$; 2°. $\operatorname{Det} \lambda = 1$.

Din 1° rezultă că $\lambda p_\alpha \lambda^{-1}$ este o matrice tot din M, deci se descompune după p_α

$$\lambda p_\alpha \lambda^{-1} = A^{\beta'}_\alpha p_{\beta'}\,, \tag{3}$$

unde $A = (A^{\beta'}_\alpha)$ este o matrice reală de ordin 4.

Cum p_α verifică (1) rezultă

$$A^{\beta'}_\alpha A^{\delta'}_\gamma g_{\beta'\delta'} = g_{\alpha\gamma} \tag{4}$$

adică matricea A aparţine grupului Lorentz omogen L (4). Relaţia (3) defineşte un homomorfism f

$$f : G \to L\,(4)\,. \tag{5}$$

Mai mult, se demonstrează că algebrele Lie ale celor două grupuri sînt izomorfe şi că

(6) $$f(G) = L(4).$$

Grupul G se mai notează Spin (4) şi se numeşte grup spinorial. Se demonstrează

T e o r e m ă. *Grupul spinorial* Spin (4) *este o acoperire de ordin* 2 *a grupului Lorentz* $L(4)$.

1.2. Notiunea de spinor. Fie $E(V_4)$ spaţiul fibrat principal al reperelor ortonormate de grup structural $L(4)$. Presupunem V_4 astfel încît din fibratul principal $E(V_4)$ se poate deduce prin extensie un fibrat principal $S(V_4)$ de grup structural Spin (4). Cînd acest lucru e posibil, un punct $z \in S(V_4)$ se va numi reper spinorial, iar $S(V_4)$ — spaţiul fibrat principal al reperelor spinoriale [4].

Dacă π este proiecţia: $S(V_4) \to E(V_4)$, un tensor din V_4 se zice raportat la un reper spinorial z dacă este raportat la reperul ortonormat $y = \pi z$.

Fie V spaţiul vectorial al matricelor complexe de tip 1×4 şi V^* dualul său.

D e f i n i ţ i a 1. Numim 1-spinor contravariant al spaţiului V_4 aplicaţia $\psi : S(V_4) \to V$, cu proprietatea

(7) $$\psi(z\lambda^{-1}) = \lambda\psi(z), \qquad \lambda \in \text{Spin}(4).$$

Punînd $\psi(z) = (\psi^a)$, (7) ne dă

(8) $$\psi^{b'} = \lambda^{b'}_{a}\, \psi^a,$$

ψ^a — componentele lui ψ în raport cu reperul spinorial z.

D e f i n i ţ i a 2. Numim 1-spinor covariant al spaţiului V_4 aplicaţia $\theta : S(V_4) \to V^*$, cu proprietatea

(9) $$\theta(z\lambda^{-1}) = \theta(z)\,\lambda^{-1}, \qquad \lambda \in \text{Spin}(4).$$

Dacă $\theta(z) = (\theta_a)$, din (9) rezultă

(10) $$\theta_{b'} = \lambda^{a}_{b'}\, \theta_a.$$

Am reamintit aici numai definiţiile spinorilor de ordin 1, dar se pot defini şi spinori de tip (p, q). Mai mult, ţinînd cont de homomorfismul canonic de la Spin (4) la $L(4)$ se pot defini şi spin-tensori de diferite tipuri [5].

De exemplu, din (3) rezultă

$$p_{\alpha'}{}^{a'}{}_{b'} = A^{\beta}_{\alpha'}\, \lambda^{a'}_{c}\, \lambda^{d}_{b'}\, p_{\beta}{}^{c}{}_{d}, \qquad A^{\beta}_{\alpha} \in L(4), \qquad \lambda \in \text{Spin}(4),$$

prin urmare $p_{\alpha}{}^{a}{}_{b}$ sînt componentele unui spin-tensor: spinor de tip (1,1) şi tensor de tip (0,1).

Spin-tensorul $p_{\alpha}{}^{a}{}_{b}$ definit de matricele Dirac p_α se numeşte spin-tensorul fundamental al spaţiului V_4.

1.3. Conexiune spinorială. D e f i n i ţ i e. Numim conexiune spinorială o conexiune infinitezimală pe fibratul principal $S(V_4)$ al reperelor spinoriale. O astfel de conexiune va fi dată de o 1-formă σ pe $S(V_4)$ cu valori în algebra Lie a grupului Spin (4).

Dacă ω este 1-forma conexiunii riemanniene pe V_4, ţinînd cont de legătura între algebrele Lie ale grupurilor Spin (4) şi $L(4)$ se demonstrează că

(11) $$\sigma^{a}_{b} = -\frac{1}{4}\,\omega^{\alpha}_{\beta}\, p_{\alpha}{}^{a}{}_{d}\, p^{\beta d}{}_{b}, \quad \text{unde} \quad p^{\beta} = g^{\beta\alpha}\, p_\alpha.$$

Fie $\omega^{\alpha}_{\beta\gamma}$ şi $\sigma^{a}_{b\rho}$ coeficienţii celor două conexiuni; din (11) rezultă

(12) $$\sigma^{a}_{b\rho} = -\frac{1}{4}\,\omega^{\alpha}_{\beta\rho}\, p_{\alpha}{}^{a}{}_{d}\, p^{\beta d}{}_{b};$$

conexiunea σ se numeşte conexiune spinorială indusă de conexiunea riemanniană ω.

După teoria generală a conexiunilor se poate introduce derivata covariantă a spinorilor şi spin-tensorilor, tensorul de curbură al conexiunii spinoriale etc. [3].

De exemplu, dacă V este simbolul derivării covariante în raport cu conexiunea σ, avem

$$\nabla_\alpha \psi^a = \partial_\alpha \psi^a + \sigma^a_{b\alpha} \psi^b , \tag{13}$$

$$\nabla_\alpha \theta_a = \partial_\alpha \theta_a - \sigma^b_{a\alpha} \theta_b , \tag{14}$$

$$\nabla_\alpha p^a_{\beta b} = \partial_\alpha p^a_{\beta b} + \sigma^a_{c\alpha} p^c_{\beta b} - \sigma^c_{b\alpha} p^a_{\beta c} - \omega^\rho_{\beta\alpha} p^a_{\rho b} , \tag{15}$$

∂ — simbolul derivatei parţiale.

Dacă $p^a_{\beta b}$ este spin-tensorul fundamental, se demonstrează că

$$\nabla_\alpha p^a_{\beta b} = 0 . \tag{16}$$

Curbura conexiunii spinoriale este dată de 2 - forma Q a lui $S(V_4)$ de tip (1,1)

$$Q^a_b = d\sigma^a_b + \frac{1}{2} \sigma^a_c \Lambda \sigma^c_b . \tag{17}$$

Dacă $Q^a_{b,\lambda\mu}$ şi $r^\alpha_{\beta,\lambda\mu}$ sînt tensorii de curbură pentru σ respectiv ω, din (12) şi (17) rezultă

$$Q^a_{b,\lambda\mu} = - \frac{1}{4} r^\alpha_{\beta,\lambda\mu} (p^a_{\alpha c} p^{\beta c}_b). \tag{18}$$

În această Notă se arată că conexiunii spinoriale σ i se poate asocia în mod unic o conexiune liniară şi se studiază o serie de proprietăţi ale acestei conexiuni.

2. Conexiunea liniară asociată unei conexiuni spinoriale

2.1. Determinarea conexiunii Γ. Fie V_4 varietatea spaţiu-timp considerată mai sus. Presupunem că pe V_4 avem şi o structură aproape produs definită de tensorul nedegenerat φ^α_β, [1], [7], deci

$$\varphi^\alpha_\beta \varphi^\beta_\gamma = \delta^\alpha_\gamma . \tag{19}$$

Tensorului φ i se poate asocia tensorul simetric covariant $\varphi_{\alpha\beta}$ dat de

$$\varphi_{\alpha\beta} = \varphi^\gamma_\alpha g_{\gamma\beta} . \tag{20}$$

Tensorul structurii aproape produs φ şi tensorul metric g verifică relaţia

$$\varphi^\rho_\alpha \varphi^\tau_\beta g_{\rho\tau} = g_{\alpha\beta} . \tag{21}$$

Cu ajutorul matricelor Dirac p_α şi a tensorului φ definim matricele h_β date de

$$h_\beta = p_\alpha \varphi^\alpha_\beta \quad \text{sau} \quad (22') \quad h^a_{\beta b} = p^a_{\alpha b} \varphi^\alpha_\beta . \tag{22}$$

T e o r e m a 1. *Matricele h_β formează un sistem de matrice Dirac.*

D e m o n s t r a ţ i e. În adevăr, avem

$$h_\beta h_\gamma + h_\gamma h_\beta = (p_\alpha \varphi^\alpha_\beta)(p_\rho \varphi^\rho_\gamma) + (p_\rho \varphi^\rho_\gamma)(p_\alpha \varphi^\alpha_\beta) = \varphi^\alpha_\beta \varphi^\rho_\gamma (p_\alpha p_\rho + p_\rho p_\alpha) =$$
$$= \varphi^\alpha_\beta \varphi^\rho_\gamma (- 2g_{\alpha\rho} e) = - 2 \varphi^\alpha_\beta \varphi^\rho_\gamma g_{\alpha\rho} e = - 2 g_{\beta\gamma} e.$$

Am obţinut astfel un nou sistem de matrice Dirac. Matricele p_α determină spin-tensorul fundamental $p_{\alpha b}^{\alpha}$ a cărui derivată covariantă în raport cu perechea de conexiuni (σ, ω) este nulă. Matricele h_α determină spin-tensorul $h_{\alpha b}^{a}$ a cărui derivată covariantă în raport cu perechea (σ, ω) nu mai este nulă ci este dată de :

$$\nabla_\beta h_{\alpha b}^{a} = \partial_\beta h_{\alpha b}^{a} + \sigma_{c\beta}^{a} h_{\alpha b}^{c} - \sigma_{b\beta}^{c} h_{\alpha c}^{a} - \omega_{\alpha\beta}^{\rho} h_{\rho c}^{a} = p_{\rho b}^{a} \overset{\circ}{\nabla}_\beta \varphi_\alpha^\rho, \tag{23}$$

unde $\overset{\circ}{\nabla}$ este simbolul derivării covariante în raport cu conexiunea riemanniană ω.

Dată conexiunea spinorială σ ne propunem să determinăm conexiunea liniară Γ, astfel încît derivata covariantă a spin-tensorului $h_\alpha^{\beta b}$ în raport cu perechea de conexiuni (σ, Γ) să fie nulă. Dacă $\Gamma_{\beta\gamma}^{\alpha}$ sînt componentele conexiunii liniare căutate, avem :

$$D_\alpha h_b^{\beta a} = \partial_\alpha h_b^{\beta a} + \sigma_{c\alpha}^{a} h_b^{\beta c} - \sigma_{b\alpha}^{c} h_c^{\beta a} - \Gamma_{\rho\alpha}^{\beta} h_b^{\rho a} = 0, \tag{24}$$

unde $h^\beta = g^{\beta\alpha} h_\alpha$.

Din (24) rezultă imediat

$$\Gamma_{\varepsilon\alpha}^{\beta} = \frac{1}{4} h_{\varepsilon a}^{b} (\partial_\alpha h_b^{\beta a} + \sigma_{c\alpha}^{a} h_b^{\beta c} - \sigma_{b\alpha}^{c} h_c^{\beta a}), \tag{25}$$

deci, conexiunea Γ cu proprietatea cerută este unic determinată. Cum h_α sînt matrice Dirac şi verifică (21), un calcul simplu ne dă conexiunea căutată sub forma

$$\Gamma_{\varepsilon\alpha}^{\beta} = \omega_{\varepsilon\alpha}^{\beta} + \varphi_\rho^\beta \overset{\circ}{\nabla}_\alpha \varphi_\varepsilon^\rho, \tag{26}$$

exprimată cu ajutorul conexiunii riemanniene ω şi a tensorului structurii aproape produs φ.

Deci, conexiunii spinoriale σ i-am asociat conexiunea liniară Γ.

2.2. Proprietăţi ale conexiunii Γ. P_1. *Conexiunea Γ coincide cu conexiunea ω dacă şi numai dacă ω este conexiune complet reductibilă* [7].

În adevăr, din (26) rezultă că Γ coincide cu ω dacă şi numai dacă $\overset{\circ}{\nabla}_\alpha \varphi_\varepsilon^\rho = 0$.

P_2. *Γ este o conexiune cu torsiune.* Dacă $T_{\beta\gamma}^{\alpha}$ sînt componentele tensorului de torsiune a conexiunii Γ, din (26) rezultă

$$T_{\beta\gamma}^{\alpha} = \varphi_\varepsilon^\alpha (\overset{\circ}{\nabla}_\gamma \varphi_\beta^\varepsilon - \overset{\circ}{\nabla}_\beta \varphi_\gamma^\varepsilon). \tag{27}$$

P_3. *Γ este o conexiune metrică,* adică

$$D_\alpha g_{\beta\gamma} = 0. \tag{28}$$

În adevăr $D_\alpha g_{\beta\gamma} = \partial_\alpha g_{\beta\gamma} - \Gamma_{\beta\alpha}^{\rho} g_{\rho\gamma} - \Gamma_{\gamma\alpha}^{\rho} g_{\beta\rho}$.

Folosind (19), (20), (21) şi faptul că $\overset{\circ}{\nabla}_\alpha g_{\beta\gamma} = 0$, un calcul simplu ne dă $D_\alpha g_{\beta\gamma} = 0$.

P_4. Fie $N_{\beta\gamma}^{\alpha}$ tensorul de torsiune al structurii aproape produs (tensorul lui Nijenhuis). Se ştie că $N_{\beta\gamma}^{\alpha}$ se exprimă cu ajutorul tensorului φ_β^α şi a derivatelor sale prin :

$$N^{\alpha}_{\beta\gamma} = \varphi^{\rho}_{\beta} (\partial_{\rho} \varphi^{\alpha}_{\gamma} - \partial_{\gamma} \varphi^{\alpha}_{\rho}) - \varphi^{\rho}_{\gamma} (\partial_{\rho} \varphi^{\alpha}_{\beta} - \partial_{\beta} \varphi^{\alpha}_{\rho}). \tag{29}$$

Din (27) şi (29) rezultă

$$\varphi^{\rho}_{\alpha} \varphi^{\beta}_{\varepsilon} N^{\alpha}_{\beta\gamma} = - 2 \Phi_3 T^{\rho}_{\varepsilon\gamma}, \tag{30}$$

legătura între tensorul de torsiune al conexiunii Γ şi tensorul de torsiune al structurii aproape produs, unde Φ_3 este operatorul Obata definit prin [6]

$$\Phi_3 T^{\rho}_{\varepsilon\gamma} = \frac{1}{2} (T^{\rho}_{\varepsilon\gamma} + \varphi^{\beta}_{\varepsilon} \varphi^{\delta}_{\gamma} T^{\rho}_{\beta\delta}). \tag{31}$$

Avem

T e o r e m a 2. *O condiţie necesară şi suficientă ca structura aproape produs să fie integrabilă este ca tensorul de torsiune T al conexiunii Γ să verifice relaţia*

$$\Phi_3 T = 0 \tag{32}$$

P_5. D e f i n i ţ i e. Spunem că perechea de conexiuni $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ este compatibilă cu structura aproape produs dată de φ, dacă transportînd prin paralelism de-a lungul unei curbe oarecare din V_4 un vector arbitrar v^{α} în raport cu conexiunea $\overset{2}{\Gamma}$, vectorul $u^{\alpha} = \varphi^{\alpha}_{\beta} v^{\beta}$ se transportă prin paralelism în raport cu conexiunea $\overset{1}{\Gamma}$ pe aceeaşi curbă [2].

Să demonstrăm că *perechea* (ω, Γ) *formează o pereche de conexiuni compatibilă cu structura aproape produs în sensul definiţiei* de mai sus.

În adevăr dacă

$$\frac{Dv^{\alpha}}{dt} = \frac{dv^{\alpha}}{dt} + \Gamma^{\alpha}_{\beta\gamma} v^{\beta} \frac{dx^{\gamma}}{dt} = 0 \tag{33}$$

şi $u^{\alpha} = \varphi^{\alpha}_{\beta} v^{\beta}$, să arătăm că

$$\frac{\overset{\circ}{\nabla} u^{\alpha}}{dt} = \frac{du^{\alpha}}{dt} + \omega^{\alpha}_{\beta\gamma} u^{\beta} \frac{dx^{\gamma}}{dt} = 0.$$

Avem

$$\frac{\overset{\circ}{\nabla} u^{\alpha}}{dt} = \frac{d(\varphi^{\alpha}_{\beta} v^{\beta})}{dt} + \omega^{\alpha}_{\beta\gamma} \varphi^{\beta}_{\rho} v^{\rho} \frac{dx^{\gamma}}{dt} = \frac{d\varphi^{\alpha}_{\beta}}{dt} v^{\beta} + \varphi^{\alpha}_{\beta} \frac{dv^{\beta}}{dt} + \omega^{\alpha}_{\beta\gamma} \varphi^{\beta}_{\rho} v^{\rho} \frac{dx^{\gamma}}{dt}.$$

Folosind (19), (26) şi (33) rezultă

$$\frac{\overset{\circ}{\nabla} u^{\alpha}}{dt} = 0 \qquad \text{c.c.t.d.} \tag{34}$$

P_6. Fie

$$\overset{m}{\Gamma}{}^{\alpha}_{\beta\gamma} = \frac{1}{2} (\Gamma^{\alpha}_{\beta\gamma} + \omega^{\alpha}_{\beta\gamma}) \tag{35}$$

conexiunea medie a perechii (Γ, ω) şi

$$\rho^{\alpha}_{\beta,\lambda\mu} = \frac{1}{2}\,[\Gamma^{\rho}_{\beta\mu}\,\omega^{\alpha}_{\rho\lambda} - \Gamma^{\rho}_{\beta\lambda}\,\omega^{\alpha}_{\rho\mu} + \omega^{\rho}_{\beta\mu}\,\Gamma^{\alpha}_{\rho\lambda} - \omega^{\rho}_{\beta\lambda}\,\Gamma^{\alpha}_{\rho\mu} + \partial_{\alpha}\,\Gamma^{\alpha}_{\beta\mu} - \tag{36}$$
$$- \partial_{\mu}\,\Gamma^{\alpha}_{\beta\lambda} + \partial_{\lambda}\,\omega^{\alpha}_{\beta\mu} - \partial_{\mu}\,\omega^{\alpha}_{\beta\lambda}\,]$$

tensorul de curbură mixtă al perechii (Γ, ω).

Un calcul simplu ne dă că *legătura între tensorii de curbură ai conexiunilor* ω *şi* $\overset{m}{\Gamma}$ *şi tensorul de curbură mixtă al perechii* (Γ, ω) *este*

$$\rho^{\alpha}_{\beta,\lambda\mu} + \Phi_1\, r^{\alpha}_{\beta,\lambda\mu} = 2\overset{m}{R}{}^{\alpha}_{\beta,\lambda\mu}\,, \tag{37}$$

unde Φ_1 — operator Obata definit prin [6]

$$\Phi_1\, r^{\alpha}_{\beta,\lambda\mu} = \frac{1}{2}(r^{\alpha}_{\beta,\lambda\mu} + \varphi^{\alpha}_{\varepsilon}\,\varphi^{\delta}_{\beta}\, r^{\varepsilon}_{\delta,\lambda\mu}). \tag{38}$$

Observaţie. Dacă conexiunea ω este complet reductibilă, atunci

$$\overset{m}{R}{}^{\alpha}_{\beta,\lambda\mu} = r^{\alpha}_{\beta,\lambda\mu} = \rho^{\alpha}_{\beta,\lambda\mu}\,.$$

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Catedra de matematică

BIBLIOGRAFIE

1. Cattaneo-Gasparini I., *Connessioni adattate a una struttura quasi prodotto.* Ann. di Mat. pura ed applicata, **63**, 133–149 (1963).
2. Cruceanu V., Miron R., *Sur les couples de connexions compatibles avec les structures presque complexes.* An. Şt. Univ. „Al. I. Cuza", Iaşi, XIII, *1*, 79–88 (1967).
3. Guy R., *Sur la dérivation covariante des spineurs.* Acad. Naz. Lincei, Sci. Fiz. Mat. Nat., Atte Ren. 8, **29**, 512 (1958).
4. Haefliger A., *Sur l'extension du groupe structural d'un espace fibré.* C. R. Acad. Sci. Paris, **243**, 558 (1956).
5. Lichnerowicz A., *Champs spinoriels et propagateurs en relativité générale.* Bull. Soc. Math. France, **92**, 11–100 (1964).
6. Obata M., *Affine Connections on Manifolds with almost Complex, Quaternion or Hermitian Structure.* Japanese J. Math., **26**, 43–77 (1956).
7. Vamanu E., *Despre structuri aproape produs şi conexiuni liniare complet reductibile asociate.* An. Şt. Univ. „Al. I. Cuza", Iaşi, XIII, *1*, 393 (1967).

CONNEXIONS SUR DES ESPACES RIEMANN PARTICULIERS

(Résumé)

Soit V_4 l'espace-temps de la rélativité générale muni d'une structure presque-produit et σ une connexion spinorielle dans V_4.

Dans cette Note on montre que à la connexion spinorielle on peut associer uniquement une connexion linéaire. On étudie quelques propriétés de cette connexion linéaire.

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ОДНО ИЗ СВОЙСТВ СТРУКТУРНЫХ ОПЕРАТОРОВ ЛУП ЛИ

ИЛИЕ БУРДУЖАН

В [1] были локально характеризованы лупы Ли при помощи векторных l-полей $X_i(u)$ структурных операторов $A^i_j(\overline{u}, u)$. С помощью l-полей мы ввели структурные функции $C^i_{jk}(u)$. Одновременно был доказан тот факт, что векторные l-поля, структурные операторы и структурные функции локальной лупы Ли удовлетворяют условиям :

$$A^i_j(\overline{u}, 0) = A^i_j(u, u) = \delta^i_j, \tag{1}$$

$$\begin{aligned} &- A^m_j(\overline{u}, u)\, A^n_k(\overline{u}, u)\, C^i_{mn}(\overline{u}) + A^m_j(\overline{u}, u)\,(X_m(\overline{u})\, A^i_k(\overline{u}, u)) - \\ &- A^m_k(\overline{u}, u)\,(X_m(\overline{u})\, A^i_j(\overline{u}, u)) + X_j(u)\, A^i_k(\overline{u}, u) - X^i_k(u)\, A^i_j(\overline{u}, u) = - \\ &= - C^m_{jk}(u)\, A^i_m(\overline{u}, u). \end{aligned} \tag{2}$$

Если $u = e$ в (2), получим

$$C^i_{kj}(\overline{u}) - C^i_{kj}(0) = \frac{\partial A^i_j(\overline{u}, 0)}{\partial u^k} - \frac{\partial A^i_k(\overline{u}, 0)}{\partial u^j}.$$

Это значит, что структурные функции, лупы, известны тогда, когда известны $C^i_{jk}(0)$ и структурные операторы $A^i_j(\overline{u}, u)$. Легко устанавливается, что $C^i_{jk}(0)$ совпадают со структурными постоянными „касательной алгебры", которая была введена Мальцевым для локальных луп Ли. Ниже мы докажем, что структурные операторы одной лупы Ли определяют и векторные l-поля лупы.

Пусть $(Q\,;., /, \backslash)$ – лупа Ли класса C^∞, $e \in Q$ – её единица и (h, U) – карта центрирована в e. Тогда имеем $h^i(e) = 0$, т.е. $(0,\ldots,0)$ – координаты единицы. Пусть теперь W – столь малая окрестность единицы лупы Q, что для всяких двух элементов x и y из W определены $x\,.\,y$, x/y, $x\backslash y$ и $x\,.\,y$, x/y, $x\backslash y \in U$. Это положение, в координатной форме, переписывается :

$$\begin{aligned} &z^i = \varphi^i(x^1,\ldots, x^r\,; y^1,\ldots, y^r), \quad z^i = \chi^i(x^1,\ldots, x^r; y^1,\ldots, y^r), \\ &z^i = \psi^i(x^1,\ldots, x^r\,; y^1,\ldots, y^r), \quad i = 1, 2,\ldots, r. \end{aligned} \tag{3}$$

Так как $x\,.\,e = x$, $e\,.\,y = y$, $x/x = x\backslash x = e$, имеем :

$$(4)\qquad \begin{gathered}\varphi^i(x^1,\dots, x^r; 0,\dots, 0) = x^i\,; \quad \varphi^i(0,\dots,0\,; y^1,\dots, y^r) = y^i\,;\\ \chi^i(x^1,\dots, x^r\,; x^1,\dots, x^r) = \psi^i(x^1,\dots, x^r\,; x^1,\dots, x^r) = 0, \quad i = 1,\dots, r.\end{gathered}$$

Из этих соотношений непосредственно следует

$$(5)\qquad \frac{\partial\varphi^i}{\partial x^j}(0,0) = \delta^i_j, \quad \frac{\partial\varphi^i}{\partial y^j}(0, 0) = \delta^i_j, \quad i, j = 1,\dots, r.$$

Соотношения (5) показывают, что вблизи e, уравнения (1_1) разрешимы относительно координат $x^1,\dots,x^r$ (соотв. $y^1,\dots,y^r$) элемента $\chi(z^1,\dots, z^r;y^1,\dots, y^r)$ (соотв. $\psi(x^1,\dots, x^r\,; z^1,\dots, z^r)$. Эти координаты являются C^∞-дифференцируемыми функциями координат y и z. Это значит, что когда локально работаем, тогда достаточно требовать только C^∞-дифференцируемость функций φ^i. Вводим вспомогательные функции:

$$u^i_j(x) = u^i_j(x^1,\dots, x^r) = \frac{\partial\varphi^i}{\partial y^j}(x^1,\dots, x^r\,;\ 0,\dots, 0).$$

Рассмотрим кривые, уравнения которых является решениями системы

$$\frac{\mathrm{d}g^i(t)}{\mathrm{d}t} = u^i_j(g\,(t))\,a^j$$

при начальных условиях $g^i(0) = 0$. Кривую соответствующую фиксированному вектору $\bar{a}\,(a^1,\dots, a^r)$ назовём геодезической кривой с направляющим вектором $\bar{a}$, является интегральная кривая l-поля, ассоциированого вектору $\bar{a}$, удовлетворяющая начальным условиям $g^i(0)=0$. Так как строение l-полей – инварианта, следует что геодезические кривые имеют инвариантный характер. Пусть Q – локальная лупа Ли и (h, U) – уже учитана карта центрирована в e. Область $V\subset U$ будет называть звездной, если вместе с каждой точкой $(x^1,\dots, x^r)$ в V имеются и все точки $(tx^1,\dots, tx^r)$ где $|\,t\,| < 1$. Карту определимую в звездной области U, будем называть канонической системой первого рода, если всякая система уравнений $x^i = a^it$ дает геодезическую кривую определенную этими уравнениями для всех значений t, при которых в U имеется точка $(a^1 t,\dots, a^rt)$.

Предложение 1. *Пусть Q – локальная лупа Ли и (h,U) – каноническая система первого рода. Всякая геодезическая кривая $g = g\,(t)$, определенная при $|t| \leqslant \alpha$ и удовлетворяющая тому условию, что $g\,(t) \in U$ при $|\,t\,| \leqslant \alpha$, в координатной форме имеет вид $g^i\,(t) = ta^i$ (при $|\,t\,| \leqslant \alpha$), где a^i – компоненты направляющего вектора $\bar{a}$ геодезической кривой g.*

Доказательство этого предложения аналогично доказательству утверждения C) из [2] стр. 297. Следует аналогичным путем доказательства теоремы 59 из [2] стр. 298, можно доказать :

Теорема 1. *Пусть $(\bar{h}, \bar{U})$ – карта центрирована в единице e лупы Ли так что закон луп, выражен в этой карте при помоши C^∞-дифференцируемых функциа. Существует тогда каноническая система первого рода (h, U) центрированная в e, так что функция $\bar{h}^{-1}\circ h|U\cap h\ (\bar{U})$ будет C^∞-дифференцируема и $\dfrac{\partial(\bar{h}^{-1}\circ h)^i}{\partial x^j}(0) = \delta^i_j$.*

Обозначим с $v^i_j(x)$ обратную матрицу матрицы $u^i_j(x)$. Как и в ([2], B) стр. 407) показывается, что (h, U) – каноническая система первого рода тогда и только тогда, когда $v^i_j(x)$ удовлетворяют условиям

$$v^i_j(x)\, x^j = x^i, \quad i, j = 1, 2,..., r. \tag{6}$$

Пусть теперь Q – лупа Ли, (h, U) – каноническая система первого рода и $v^i_j(x)$ – выражение вспомогательных функций в этой карте. Положим

$$w^i_j(t) = w^i_j(t, \bar{a}) = tv^i_j(ta^1,..., ta^r) = tv^i_j(t\bar{a}), \quad i, j = 1, 2,..., r, \tag{7}$$

где a – фиксированный вектор, а $t\bar{a}$ обозначает элемент с координатами ta^i.

Тогда имеют место следующие соотношения :

$$v^i_j(x) = w^i_j(1, x), \tag{8}$$

$$w^i_j(0, \bar{a}) = 0, \tag{9}$$

$$\frac{\mathrm{d}w^i_j(t)}{\mathrm{d}t} = \delta^i_j + C^i_{kl}(ta)\, a^k w^l_j\,(t), \quad i, j, k, l = 1, 2,.., r. \tag{10}$$

Соотношения (8) и (9) очевидны, а соотношение (10) доказывается так же и соотношение (29) в [2], стр. 408.

Считаем теперь $C'^i_{jk}(x)$ – выражения структурных функций лупы Ли в данной карте (h, U) (центрирована в e). Они удовлетворяют соотношением

$$C'^i_{jk}(x) = -\, C'^i_{kj}(x), \quad i, j, k = 1, 2,..., r. \tag{11}$$

Тождеству Якоби и ряду следствий тождества Якоби. Положим

$$\bar{h}'^i_{i_3 i_2 i_1} = C^i_{i_3 s} C^s_{i_2 i_1} + C^i_{i_2 s} C^s_{i_1 i_3} + C^i_{i_1 s} C^s_{i_3 i_2} - X_{i_3} C^i_{i_2 i_1} - X_{i_2} C^i_{i_1 i_3} - X_{i_1} C^i_{i_3 i_2} \tag{12}$$

и, индуктивно, определим функции

$$\bar{h}'^i_{i_n \ldots i_2 i_1} = X_{i_n} \bar{h}'^i_{n-1 \ldots i_2 i_1} - C^i_{i_n j} h'^j_{n-1 \ldots i_2 i_1}. \tag{13}$$

Тогда тождество Якоби и его следствия $[X_{i_n}, [...[X_{i_4}, [X_{i_3}, [X_{i_2}, X_{i_1}]] + [X_{i_2}, [X_{i_1}, X_{i_3}]] + [X_{i_1}, [X_{i_3}, X_{i_2}]]] ...] = 0$ имеют вид:

$$\bar{h}'^i_{i_n \ldots i_2 i_1}(x) = 0\,, \quad n \geqslant 3. \tag{14}$$

Очевидно, имеем и соотношения

$$\bar{h}'^i_{i_n \ldots i_2 i_1}(0) = 0, \quad n \geqslant 3. \tag{15}$$

Считаем теперь выражения $C'^i_{jk}(x)$ – структурных функций лупы в канонической системе координат первого рода, ассоциированной карте (h, U). Согласно теореме 1, имеем:

$$C'^i_{jk}(0) = C^i_{jk}(0), \quad \frac{\partial^\alpha C'^i_{jk}(0)}{\partial\,(x^1)^{\alpha_1} \ldots \partial\,(x^r)^{\alpha_r}} = \frac{\partial C^i_{ik}(0)}{\partial\,(x^1)^{\alpha_1} \ldots \partial\,(x^r)^{\alpha_r}}\,, \tag{16}$$

$$\alpha = \alpha_1 + \ldots + \alpha_r, \quad \alpha_i \geqslant 0, \quad i = 1, 2,..., r.$$

Можно ввести и в эту карту выражения определенные функций в (12) и их обозначаем $\overline{h}'^{i}_{i_n \dots i_2 i_1}(x)$. Очевидно имеем:

$$\overline{h}'^{i}_{i_n \dots i_2 i_1}(0) = \overline{h}^{i}_{i_n \dots i_2 i_1}(0) = 0. \tag{17}$$

Нас интересует теперь следующий вопрос. Пусть $C^i_{jk}(x)$ – выражение структурных функций лупы Ли в канонической системе координат первого рода. Тогда имеем:

$$\overline{h}^{i}_{i_n \dots i_2 i_1}(0) = 0, \quad n \geqslant 3, \tag{10'}$$

$$C^i_{jk}(0) = -C^i_{kj}(0), \quad i, j, k, i_1, \dots, i_n = 1, 2, \dots, r. \tag{13'}$$

Рассмотрим систему обыкновенных дифференциальных уравнений,

$$\frac{\mathrm{d}w^i_j}{\mathrm{d}t} = \delta^i_j + C^i_{kl}(ta)\, a^k w^l_j, \tag{17}$$

где a – это фиксированный вектор, а w^i_i неизвестные функции параметра t. Система (17) линейная, и поэтому ее решение C^∞ -дифференцируемо. Обозначим через $w^i_j(t, a)$ решение системы (17) с начальным условием

$$w^i_j(0, a) = 0. \tag{18}$$

Положим далее

$$v^i_j(x) = w^i_j(1, x). \tag{19}$$

Мы хотим доказать, что функции $v^i_j(x)$ удовлетворяют соотношениям:

$$v^i_j(0) = \delta^i_j, \tag{20}$$

$$\frac{\partial v^i_k(x)}{\partial x^j} - \frac{\partial v^i_j(x)}{\partial x^k} = C^i_{ls}(x)\, v^l_j(x)\, v^s_k(x), \tag{21}$$

$$v^i_j(x)\, x^j = x^i. \tag{22}$$

Для установления этих соотношений действуем так же как в доказательстве теоремы 87 из [2], стр. 409. Положим

$$h^i(t) = w^i_j(t, a)\, a^j - ta^i. \tag{23}$$

Очевидно, $h^i(t) \equiv 0$ и непосредственно установим

$$\frac{\mathrm{d}h^i}{\mathrm{d}t} = C^i_{jk}(ta)\, a^j h^k.$$

Получим тогда $h^i(t) \equiv 0$. Это значит что

$$w^i_j(t, a)\, a^j = ta^i \tag{24}$$

и $h^i(1) \equiv 0$ даст нам соотношение (22). Рассмотрим теперь функции

$$h^i_j(\tau) = w^i_j(t\tau, a) - tw^i_j(\tau, ta). \tag{25}$$

Вычислим производную функций $h^i_j(\tau)$. Получаем, принимая во внимание соотношение (17):

$$\frac{\mathrm{d}h^i_j(\tau)}{\mathrm{d}\tau} = tC^i_{kl}(t\tau, a)\, a^k h^l_j(\tau)\,.$$

Потому что $h^i_j(0) = 0$ следует что $h^i_j(\tau) \equiv 0$, т.е.

$$w^i_j(t\tau, a) = tw^i_j(\tau, ta). \tag{26}$$

Следовательно $w^i_j(t, 0) = tw^i_j(1, 0) = t\delta^i_j$,, т.е. $w^i_j(1, 0) = \delta^i_j$. Это значит, что для столь малово значения параметра t $(t \neq 0)$, $w^i_j(t, a)$ имеет обратную матрицу т.е. $\overline{w}^i_j(t, a)$ имеет обратную матрицу. Имеет $\overline{w}^i_j(t, a) = \frac{1}{t}\overline{w}^i_j(1, ta)$, или

$$\overline{w}^i_j(1, ta) = t\overline{w}^i_j(t, a), \tag{27}$$

где $\overline{w}^i_j$ значит $(w^i_j)^{-1}$. Строим теперь вспомогательные функции

$$h^i_{i_2 i_1}(t) = \frac{\partial w^i_{i_1}(t, a)}{\partial a^{i_2}} - \frac{\partial w^i_{i_2}(t, a)}{\partial a^{i_1}} - C^i_{jk}(ta)\, w^j_{i_2}(t, a)\, w^k_{i_1}(t, a), \tag{28}$$

$$\begin{aligned} h^i_{i_3 i_2 i_1}(t) &= C^i_{i_3 j}(ta)\, C^i_{i_2 i_1}(ta) + C^i_{i_2 j}(ta)\, C^j_{i_1 i_3}(ta) + C^i_{i_1 j}(ta)\, C^j_{i_3 i_2}(ta) - \\ &- \overline{w}^j_{i_3}(1, ta)\frac{\partial C^i_{i_2 i_1}(ta)}{\partial a^j} - \overline{w}^j_{i_2}(1, ta)\frac{\partial C^i_{i_1 i_3}(ta)}{\partial a^j} - \overline{w}^j_{i_1}(1, ta)\frac{\partial C^i_{i_3 i_2}(ta)}{\partial a^j} \end{aligned} \tag{29}$$

а потом, индуктивно, функции

$$h^i_{i_n \ldots i_2 i_1}(t) = \overline{w}^k_{i_n}(t, a)\, \frac{\partial h^i_{i_{n-1} \ldots i_2 i_1}}{\partial a^k} - C^i_{i_n k}(ta)\, h^k_{i_{n-1} \ldots i_2 i_1}. \tag{30}$$

Скажем, что функция $f(t, a)$ имеет своиство А, если она выражается как сумма произведений, имеющих как факторы функции $C^i_{jk}(ta)$, $\overline{w}^i_j(1, ta)$ или их производные. В силу определения, индуктивно установляется функции $h^i_{i_n \ldots i_2 i_1}$ имеют свойство А. Тогда получим:

$$\frac{\mathrm{d}h^i_{i_n \ldots i_2 i_1}}{dt} = \frac{1}{t}\frac{\partial h^i_{i_n \ldots i_2 i_1}}{\partial a^j}\, a^j, \qquad n \geqslant 3. \tag{31}$$

Из соотношения (30) получим:

$$h^i_{i_{n+1} i_n \ldots i_2 i_1}\, a^{i_{n+1}} = \overline{w}^k_{i_{n+1}}(1, ta)\, a^{i_{n+1}}\left(\frac{1}{t}\,\frac{\partial h^i_{i_n \ldots i_2 i_1}}{\partial a^k}\right) - C^i_{i_{n+1} k}\, a^{i_{n+1}} h^k_{i_n \ldots i_1}(t) =$$

$$= \frac{1}{t}\, a^k\, \frac{\partial h^i_{i_n \ldots i_2 i_1}}{\partial a^k} - C^i_{i_{n+1} k} a^{i_{n+1}} h^k_{i_n \ldots i_2 i_1}(t) = \frac{\mathrm{d}h^i_{i_n \ldots i_2 i_1}}{\mathrm{d}t} - C^i_{i_{n+1} k} a^{i_{n+1}} h^k_{i_n \ldots i_2 i_1}(t).$$

Следовательно имеем:

$$\frac{\mathrm{d}h^i_{i_n\ldots i_2 i_1}}{\mathrm{d}t} = C^i_{i_{n+1}k}(ta)\, a^{i_{n+1}} h^k_{i_n\ldots i_2 i_1}(t) + h^i_{i_{n+1} i_n\ldots i_2 i_1} a^{i_{n+1}}. \tag{32}$$

Вычислим теперь производную функции $h^i_{i_2 i_1}(t)$. Принимая в внимание соотношения (17) и (24) мы получаем:

$$\frac{\mathrm{d}h^i_{i_2 i_1}}{\mathrm{d}t} = \frac{\partial^2 w^i_{i_1}(t,a)}{\partial t\, \partial a^{i_2}} - \frac{\partial^2 w^i_{i_2}(t,a)}{\partial t\, \partial a^{i_1}} - \frac{\partial C^i_{jk}(ta)}{\partial a^l} a^l w^j_{i_2}(t,a)\, w^k_{i_1}(t,a) -$$

$$- C^i_{jk}(ta) \frac{\partial w^j_{i_2}(t,a)}{\partial t} w^k_{i_1}(t,a) - C^i_{jk}(ta)\, w^j_{i_2}(t,a) \frac{\partial w^k_{i_1}(t,a)}{\partial t} \equiv$$

$$\equiv t \frac{\partial C^i_{jk}(ta)}{\partial a^{i_2}} a^j w^k_{i_1}(t,a) + C^i_{i_2 j}(ta)\, w^j_{i_1}(t,a) + C^i_{jk}(ta)\, a^j \frac{\partial w^k_{i_1}(t,a)}{\partial a^{i_2}} -$$

$$- t \frac{\partial C^i_{jk}(ta)}{\partial a^{i_1}} a^j w^k_{i_2}(t,a) - C^i_{jk}(ta)\, a^j \frac{\partial w^k_{i_2}(t,a)}{\partial a^{i_1}} - C^i_{i_1 j}(ta)\, w^j_{i_2}(t,a) -$$

$$- \frac{\partial C^i_{jk}(ta)}{\partial a^s} a^s w^j_{i_2}(t,a)\, w^k_{i_1}(t,a) - C^i_{i_2 s}(ta)\, w^s_{i_1}(t,a) - C^i_{j i_1}(ta)\, w^j_{i_2}(t,a) -$$

$$- C^i_{jk}(ta)\, C^j_{ls}(ta)\, a^l w^s_{i_2}(t,a)\, w^k_{i_1}(t,a) - C^i_{js}(ta)\, C^s_{kl}(ta)\, w^j_{i_2}(t,a)\, w^l_{i_1}(t,a).$$

Это возможно писать в виде:

$$\frac{\mathrm{d}h^i_{i_2 i_1}}{\mathrm{d}t} = C^i_{kj}(ta)\, a^k h^j_{i_2 i_1} + h^i_{jkl}\, a^j w^k_{i_2}(t,a)\, w^l_{i_1}(t,a). \tag{33}$$

Соотношения (32) и (33) показывают, что функции $h^i_{i_n\cdots i_2 i_1}$, $n \geqslant 2$ являются решениями определенного линейного однородного дифференциального уравнения первого порядка, удовлетворяющими начальным условиям (16). Тогда, согласно теоремы из [2], стр. 153, следует что $h^i_{i_n\cdots i_2 i_1}(t) \equiv 0$, т.е. $h^i_{i_n\cdots i_2 i_1}(1) \equiv 0$. Это значит, что функции $v^i_j(x) = w^i_j(1, x)$ удовлетворяют соотношениям (21). Итак, наш вопрос решен.

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Ясски политехнически институт,
Кафедра математики

ЛИТЕРАТУРА

1. Burdujan I., *Asupra algebrei Lie a cîmpurilor de vectori ale unui loop Lie.* Ann. şt. Univ. „Al. I. Cuza", Iaşi, Sect. I, Mat., XX, 107–121 (1974).
2. Hogbe-Nlend H., *Théorie des Bornologies et Applications.* Lecture Notes in Mathematics, Nr. *213*, Springer-Verlag, 1971.
3. Понтрягин Л. С., *Непрерывные группы.* Изд. „Наука", М., 1973.

O PROPRIETATE A OPERATORILOR DE STRUCTURĂ AI UNUI LOOP LIE

(Rezumat)

În această notă se stabileşte faptul că operatorii de structură ai unui loop Lie determină local *l*-cîmpurile loopului.

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ASUPRA UNEI CLASE DE ALGEBRE NEASOCIATIVE

DE

I. ENESCU

1. Considerînd algebra A, comutativă cu scalari reali, al cărei produs este dat prin tabela

(1)

	e_1	e_2	e_3
e_1	0	e_3	e_2
e_2	e_3	0	e_1
e_3	e_2	e_1	0

se constată imediat că această algebră este neasociativă şi nu este cu puteri asociative, nici alternativă şi nici Jordan. În schimb, în algebra A are loc proprietatea

$$x^2 \,.\, x^2 = 8\, abcx, \tag{2}$$

oricare ar fi elementul $x = ae_1 + be_2 + ce_3$ din A.

Specific algebrei A, este faptul că ea are o bază formată din nilpotenţi de indice doi, cu proprietatea că, produsul oricăror două elemente ale acestei baze este egal cu un singur alt element din bază diferit de cele două care s-au înmulţit :

$$e_i\, e_j = e_j\, e_i = \begin{cases} e_k & i \neq j \\ 0 & i = j \end{cases} \qquad i \neq k,\ j \neq k\,; \quad i, j,\ k = \overline{1, 3}. \tag{3}$$

Vom spune că algebra A are proprietatea P ; se vede imediat că această proprietate este exprimată prin (3).

Modul de construcţie al algebrei A, sugerează generalizarea acestei structuri pentru dimensiuni superioare lui trei. Dată fiind complexitatea problemei, vom prezenta, în această notă, un procedeu de construcţie a unor algebre cu proprietatea P, de ordin patru şi respectiv cinci, algebre pentru care se vor pune în evidenţă şi unele proprietăţi de structură.

2. Să considerăm algebra de ordin patru, dată prin tabela

(4)

	e_1	e_2	e_3	e_4
e_1	0	e_3+e_4	e_2+e_4	e_2+e_3
e_2	e_3+e_4	0	e_1+e_4	e_1+e_3
e_3	e_2+e_4	e_1+e_4	0	e_1+e_2
e_4	e_2+e_3	e_1+e_3	e_1+e_2	0

Vom obţine două algebre de ordin patru, cu proprietatea P, considerînd în (4) numai cîte un termen din respectiv fiecare produs $e_i e_j$.

Notînd cu B_1 şi B_2 aceste două algebre, ele sînt date prin respectiv tabelele :

(5)

	u_1	u_2	u_3	u_4
u_1	0	u_3	u_2	u_2
u_2	u_3	0	u_1	u_1
u_3	u_2	u_1	0	u_1
u_4	u_2	u_1	u_1	0

(6)

	v_1	v_2	v_3	v_4
v_1	0	v_4	v_4	v_3
v_2	v_4	0	v_4	v_3
v_3	v_4	v_4	0	v_2
v_4	v_3	v_3	v_2	0

Schimbînd baza în (6) după : $\varepsilon_1 = v_4$, $\varepsilon_2 = v_3$, $\varepsilon_3 = v_2$, $\varepsilon_4 = v_1$, se obţine o tabelă identică cu (5), ceea ce stabileşte

T e o r e m a 1. *Algebrele B_1 şi B_2 sînt izomorfe şi, ele, se pot reprezenta prin tabela* (5).

Se vede că algebra (5) are un ideal de ordin trei izomorf cu A. Mai mult, produsul oricăror două elemente din (5) este conţinut în idealul A. De asemenea, la fel cu A, algebra (5) este neasociativă şi nu este alternativă, nici cu puteri asociative şi nici Jordan.

Fie acum

$$x = \alpha u_1 + \beta u_2 + \gamma u_3 + \delta u_4, \tag{7}$$

un element arbitrar din B_1. Calculînd produsul $xx = x^2$, după tabela (5), se obţine $y = x^2 = au_1 + bu_2 + cu_3$, unde

$$a = 2(\beta\gamma + \beta\delta + \gamma\delta), \quad b = 2\alpha(\gamma + \delta), \quad c = 2\alpha\beta. \tag{8}$$

Întrucît elementul y aparţine idealului A, el satisface relaţia (2), dovedind astfel

T e o r e m a 2. *Orice element x al algebrei B_1 satisface relaţia*

$$(x^2)^2 . (x^2)^2 = 8\, abcx^2, \tag{9}$$

unde a, b, c se exprimă prin (8) *cu semnificaţia din* (7).

3. În cazul dimensiunii cinci, fie algebra

(10)

	e_1	e_2	e_3	e_4	e_5
e_1	0	$e_3+e_4+e_5$	$e_2+e_4+e_5$	$e_2+e_3+e_5$	$e_2+e_3+e_4$
e_2	$e_3+e_4+e_5$	0	$e_1+e_4+e_5$	$e_1+e_3+e_5$	$e_1+e_3+e_4$
e_3	$e_2+e_4+e_5$	$e_1+e_4+e_5$	0	$e_1+e_2+e_5$	$e_1+e_2+e_4$
e_4	$e_2+e_3+e_5$	$e_1+e_3+e_5$	$e_1+e_2+e_5$	0	$e_1+e_2+e_3$
e_5	$e_2+e_3+e_4$	$e_1+e_3+e_4$	$e_1+e_2+e_4$	$e_1+e_2+e_3$	0

Obţinem trei algebre C_1, C_2, C_3 de ordin cinci, cu proprietatea P, considerînd în (10), ca şi în cazul patru-dimensional, numai cîte un termen din respectiv fiecare produs $e_i e_j$. În acest mod, algebrele C_1, C_3, C_3 sînt date prin respectiv tabelele :

(11)

	u_1	u_2	u_3	u_4	u_5
u_1	0	u_3	u_2	u_2	u_2
u_2	u_3	0	u_1	u_1	u_1
u_3	u_2	u_1	0	u_1	u_1
u_4	u_2	u_1	u_1	0	u_1
u_5	u_2	u_1	u_1	u_1	0

,

(12)

	v_1	v_2	v_3	v_4	v_5
v_1	0	v_4	v_4	v_3	v_3
v_2	v_4	0	v_4	v_3	v_3
v_3	v_4	v_4	0	v_2	v_2
v_4	v_3	v_3	v_2	0	v_2
v_5	v_3	v_3	v_2	v_2	0

,

(13)

	w_1	w_2	w_3	w_4	w_5
w_1	0	w_5	w_5	w_5	w_4
w_2	w_5	0	w_5	w_5	w_4
w_3	w_5	w_5	0	w_5	w_4
w_4	w_5	w_5	w_5	0	w_3
w_5	w_4	w_4	w_4	w_3	0

.

Efectuînd în (13) schimbarea de bază : $\varepsilon_1 = w_5$, $\varepsilon_2 = w_4$, $\varepsilon_3 = w_3$, $\varepsilon_4 = w_1$, $\varepsilon_5 = w_2$, se obţine o tabelă identică cu (11), fapt care arată că algebrele C_1 şi C_2 sînt izomorfe.

Se pune, în mod natural, întrebarea dacă şi algebra C_2 este izomorfă cu C_1 şi C_3. Răspunsul este negativ, aşa cum se constată, observînd că în algebra C_1 au loc relaţiile $u_i u_j = u_1$, $i \neq j$; $i, j = \overline{2, 5}$, în timp ce, în algebra C_2 oricare patru nilpotenţi liniar independenţi nu au această proprietate. Am demonstrat astfel

T e o r e m a 3. *Există două tipuri neizomorfe de algebre C_i, $i = \overline{1, 3}$, şi ele se pot reprezenta prin respectiv tabelele* (11) *şi* (12).

Dacă în algebra C_2 schimbăm baza după : $\varepsilon_1 = v_1$, $\varepsilon_2 = v_3$, $\varepsilon_3 = v_2$, $\varepsilon_4 = v_1$, $\varepsilon_5 = v_5$, obţinem tabela

(14)

	ε_1	ε_2	ε_3	ε_4	ε_5
ε_1	0	ε_3	ε_2	ε_2	ε_3
ε_2	ε_3	0	ε_1	ε_1	ε_3
ε_3	ε_2	ε_1	0	ε_1	ε_2
ε_4	ε_2	ε_1	ε_1	0	ε_2
ε_5	ε_3	ε_3	ε_2	ε_2	0

.

Se vede direct din (11) şi respectiv (14), că algebrele C_1 şi C_2 conţin ca ideal, algebra B_1 dată prin (5). La rîndu-i, aceasta din urmă are ca ideal algebra A. Evident, idealul A este minimal pentru algebrele C_1 şi C_2.

Fie $x = \sum_1^5 \alpha_i u_i$ un element arbitrar din C_1. După (11) avem $x^2 = au_1 + bu_2 + cu_3$, unde

$$(15)\quad \begin{cases} a = 2\,(\alpha_2\,\alpha_3 + \alpha_2\,\alpha_4 + \alpha_2\,\alpha_5 + \alpha_3\,\alpha_4 + \alpha_3\,\alpha_5 + \alpha_4\,\alpha_5), \\ b = 2\,(\alpha_1\,\alpha_3 + \alpha_1\,\alpha_4 + \alpha_1\,\alpha_5), \\ c = 2\,\alpha_1\,\alpha_2\,. \end{cases}$$

La fel, dacă $x = \sum_1^5 \beta_i\,\varepsilon_i$ este arbitrar în C_2, utilizînd (14) avem $x^2 = au_1 + bu_2 + cu_3$, unde, de data aceasta,

$$(16)\quad \begin{cases} a = 2\,(\beta_2\,\beta_3 + \beta_2\,\beta_4 + \beta_2\,\beta_4), \\ b = 2\,(\beta_1\,\beta_3 + \beta_1\,\beta_4 + \beta_3\,\beta_5 + \beta_4\,\beta_5), \\ c = 2\,(\beta_1\,\beta_2 + \beta_1\,\beta_5 + \beta_2\,\beta_5). \end{cases}$$

Cum, în ambele algebre considerate elementul x^2 aparţine idealului A, rezultă

T e o r e m a 4. *Oricare element din respectiv fiecare algebră C_1, C_2, satisface relaţia* (9), *în care a, b, c sînt daţi de respectiv expresiile* (15) *şi* (16).

4. Deşi nu este exhaustiv, procedeul prin care am obţinut algebre cu proprietatea P a condus la algebre care au ca ideal minimal algebra A. În consecinţă, deşi neizomorfe, algebrele C_1 şi C_2 au proprietatea de a conţine algebra B_1 ca ideal de ordin patru. Mai mult, toate algebrele cercetate, satisfac identitatea (9).

Printre problemele deschise care rămîn, semnalăm pe aceea care constă în determinarea tuturor tipurilor neizomorfe de algebre de ordin patru şi respectiv cinci, care, avînd proprietatea P, au şi proprietatea (9).

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Institutul politehnic, Iaşi,
Catedra de matematici

SUR UNE CLASSE D'ALGÈBRES NONASSOCIATIVES

(Résumé)

En considérant la propriété P, exprimée par (3), de l'algèbre (1), on construit certaines algèbres d'ordre quatre et respectivement cinq qui ont la même propriété P. On détermine les types non isomorphes de cettes algèbres, en mettant aussi en évidence certaines propriétés de structure.

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D.C.

ON STABILITY OF NON-LINEAR DIFFERENTIAL SYSTEMS

BY

PAVEL TALPALARU

1. Introduction. Many papers in differential equations have studied the stability and asymptotic stability of the non-linear time-varying diferential equations of the form

$$\dot{x} = A(t)\,x + g(t, x), \tag{1.1}$$

where x, f are real n-dimensional vectors, $A(t)$ is an $n \times n$ real matrix and $g(t, x)$ is a continuous function of t and x. One of the frequently used method in studying stability problems is the Lyapunov direct method but there is no systematic technique for the construction of Lyapunov functions. C. V. Pao [4] has considered a different approach, motived by the concept of an equivalent inner product and a dissipative operator, which not only maintains the advantage of the Lyapunov direct method in the study of the stability but also offers a direct technique in estimating the stability region without relying on the construction of the Lyapunov function. Similar results have been given earlier also by B. P. Demidovič [1], [2].

In this paper we will extend some of the results of C. V. Pao to the non-linear systems of the form

$$\dot{x} = f(t, x) + g(t, x), \tag{1.2}$$

by using the method of B. Demidovič. We shall establish sufficient conditions to insure the stability and asymptotic stability of a solution.

2. Preliminary Results. Let R^n be the n-dimensional Euclidean space. For any two column vectors $x = (x_1, \ldots, x_n)'$, $y = (y_1, \ldots, y_n)'$ in R^n the inner product is defined by

$$< x, y > = x^* y = \sum_{i=1}^{n} x_i\, y_i\,, \tag{2.1}$$

where a star denotes the transpose of a vector in R^n. The norm induced by the inner product $< \cdot, \cdot >$ is given by $\| x \| = \sqrt{< x, x >}$. Throughout this paper, we shall reserve $< \cdot, \cdot >$ as the inner product defined by (2.1) and $\| \cdot \|$ as the norm of the Hilbert space $X = (R^n, < \cdot, \cdot >)$.

To establish our results we need the following three lemmas.

Lemma 2.1 ([4]). Let V be a real symmetric positive definite matrix. Then the inner product $< \cdot, \cdot >$ defined by

$$< x, y >_v = < x, Vy >, \quad x, y \in R^n \tag{2.2}$$

is equivalent to $< \cdot, \cdot >$.

Lemma 2.2 ([4]). Let $< \cdot, \cdot >_v$ be the inner product defined by (2.2). If $z(t)$ is differentiable in the interval $[t_0, \infty)$ then $\| z(t) \|_v^2$ is differentiable and

$$\frac{d}{dt} \| z(t) \|_v^2 = 2 < \frac{d}{dt} z(t), z(t) >_v, \quad t \in [t_0, \infty), \tag{2.3}$$

where $\| \cdot \|_v$ is the norm induced by $< \cdot , \cdot >$.

A more general formulation of Lemma 2.2 has been given by T. Kato [3].

Lemma 2.3 ([1]). Assume that $x = (x_1, \ldots, x_n) \in R^n$, $\Omega \subset R^n$ and $f(t, x) = (f_1(t, x), \ldots, f_n(t,x))$ belongs to $C^1([t_0, \infty) \times \Omega)$. Let us denote through $\lambda(t, x)$ and $\Lambda(t, x)$ respectively the smallest and the largest characteristic numbers of a symmetrised Jacobi's matrix

$$J_s(t, x) = \frac{1}{2}\left[f'_x(t, x) + f'^*_x(t, x) \right] = \frac{1}{2}\left[\frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i} \right]. \tag{2.4}$$

Then for any points (t, x) and $(t, x + h)$ such that $\xi = x + \tau h \in \Omega$ with $\tau \in [0, 1]$ an inequality

$$\lambda_m < h, h > \leqslant < f(t, x + h) - f(t, x), h > \leqslant \Lambda_M < h, h >, \tag{2.5}$$

is fulfilled, where $\lambda_m = \min_{\xi} \lambda(\xi)$ and $\Lambda_M = \max_{\xi} \Lambda(\xi)$.

3. Main Results. In this section we shall investigate the non-linear system (1.2). It is known that if $f(t, x)$ and $g(t, x)$ are continuous on $\{t_0 \leqslant t < \infty, \| x \| < \infty\}$ and if, in addition, they are Lipschitz continuous in x, then given any initial value $x_0 \in R^n$ there exists a unique solution $x = x(t)$ such that $x(t_0) = x_0$. Actually the Lipschitz continuity can be weakened to some extent. Because of its close relationship with the stability problem we first state a uniqueness result by weakening the Lipschitz condition.

Theorem 3.1. *Assume that $f(t, x)$ and $g(t, x)$ are continuous for $t_0 \leqslant t < \infty$, $\| x \| < \infty$. If there exists a continuous function $k(t)$ on $[t_0, \infty)$ such that for any $x, y \in X$, $t \in [t_0, \infty)$*

$$< g(t, x) - g(t, y), x - y > \leqslant k(t) \| x - y \|^2, \tag{3.1}$$

and if, in addition, there exists continuous functions $f'_x(t, x)$ and $\gamma(t)$ such that

$$\Lambda(f'_x(t, x)) \leqslant \gamma(t), \quad \text{for } t \in [t_0, \infty) \text{ and } \| x \| < \infty, \tag{3.2}$$

then given any initial value x_0 there exists a unique solution $x = x(t)$ with $x(t_0) = x_0$. Moreover, for any two solutions $x(t)$ and $y(t)$ with $x(t_0) = x_0$, $y(t_0) = y_0$

$$\| x(t) - y(t) \| \leqslant \| x_0 - y_0 \| \exp\left(\int_{t_0}^{t} (\gamma(s) + k(s))\, ds \right). \tag{3.3}$$

Proof. Let $x(t)$, $y(t)$ be any two solutions of (1.2) with $x(t_0) = x_0$, $y(t_0) = y_0$ and let $z(t) = x(t) - y(t)$. Then $\dot{z}(t) = f(t, x) - f(t, y) + g(t, x) - g(t, y)$ and $z(t_0) = x_0 - y_0$. By Lemma 2.2

$$\frac{d}{dt} \| z(t) \|^2 = 2 < f(t, x) - f(t, y), z(t) > + 2 < g(t, x) - g(t, y), z(t) > =$$
$$= 2 < f(t, x) - f(t, y), z(t) > + 2k(t) \| z(t) \|^2.$$

By Lemma 2.3 we have

$$2 < f(t, x) - f(t, y), x - y > \leqslant 2 \Lambda (f'_x(t, x)) \| x - y \|^2 \leqslant 2\gamma(t) \| z(t) \|^2.$$

Therefore,

$$\frac{d}{dt} \| z(t) \|^2 \leqslant 2 (\gamma(t) + k(t)) \| z(t) \|^2,$$

from where

$$\| z(t) \| \leqslant \| z(t_0) \| \exp\left(\int_{t_0}^{t} (\gamma(s) + k(s))\, ds\right),$$

that is relation (3.3).

To show the uniqueness of a solution, let $x_0 = y_0$. Since both $\gamma(t)$ and $k(t)$ are bounded on any finite interval $[t_0, T]$ it follows from (3.3) that $\| x(t) - y(t) \| = 0$ for all $t \in [t_0, T]$. Since T can be arbitrarly it follows that $\| x(t) - y(t) \| = 0$ for $t \in [t_0, \infty)$. Hence the theorem is proved.

Remarks. 1. When $f(t, x) = A(t)x$, where $A(t)$ is a continuous on the interval $[t_0, \infty)$ matrix, the condition (3.2) becomes $\Lambda(A(t)) \leqslant \gamma(t)$, where $\Lambda(A(t))$ is the largest eigenvalue of $\frac{1}{2}[A(t) + A^*(t)]$.

2. If $g(t, x)$ is Lipschitz continuous in x with a non-negative on $[t_0, \infty)$ Lipschitz function $k(t)$, then (3.1) is satisfied since

$$< g(t, x) - g(t, y), x - y > \leqslant \| g(t, x) - g(t, y) \| \, \| x - y \| \leqslant$$
$$\leqslant k(t) \| x - y \|^2.$$

3. From the inequality (3.3) it follows that if $\lim\limits_{t \to \infty} \int_{t_0}^{t} (\gamma(s) + k(s))\, ds < \infty$ (or $= -\infty$), then any solution of (1.2) is stable (asymptotically stable).

It is known that in studying the stability of an equilibrium solution x_e we can always assume that $x_e = 0$. Moreover, condition (3.1) can be weakened to some extent. For the sake of convenience and to insure the existence of a solution, we make the following *assumption* (A):

a) $f(t, x)$ and $g(t, x)$ are defined and continuous for $t_0 \leqslant t < \infty$, $\| x \| < \infty$ and there exists $f'_x(t, x)$ continuous on the same set;

b) $f(t, 0) = g(t, 0) = 0$ and there exists a continuous function $k(t)$ on $[t_0, \infty)$ such that

$$< g(t, x), x > \leqslant k(t) \| x \|^2, \quad \text{for} \quad x \in X, \quad t \in [t_0, \infty). \tag{3.4}$$

The stability and asymptotic stability of a solution in the following theorems are in the sense of global stability, that is, the stability region is the whole space X.

Theorem 3.2. *Assume that assumption (A) is satisfied. If there exist non-negative continuous functions $\beta(t)$ and $\gamma(t)$ such that for some $T > t_0$*

$$\Lambda (f'_x(t, x)) \leqslant \gamma(t), \quad \textit{for} \quad t \in [t_0, T], \quad x \in X, \tag{3.5}$$

$$\Lambda\,(f'_x\,(t,\,x)) \leqslant -\,\beta\,(t), \quad \textit{for} \quad t \in [T,\,\infty), \quad x \in X, \tag{3.6}$$

with $\beta\,(t) \geqslant k\,(t)$ *for* $t \in [T,\,\infty)$ *then the null solution is stable. If, in addition,*

$$\lim_{t\to\infty} \int_T^t (\beta\,(s) - k\,(s))\,\mathrm{d}s = \infty \tag{3.7}$$

it is assymptotically stable.

Proof. By the assumption (A), given any initial value x_0 there exists a solution of (1.2) $x = x\,(t)$ such that $x\,(t_0) = x_0$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \| x\,(t) \|^2 = 2 < f\,(t,\,x\,(t)) + g\,(t,\,x\,(t))\,(x) > = \tag{3.8}$$

$$= 2 < f\,(t,\,x\,(t)),\,x\,(t) > + 2 < g\,(t,\,x\,(t)),\,x(t) >, \quad t \in [t_0\,,\infty).$$

Since

$$< f\,(t,\,x\,(t)),\,x\,(t) > = < f\,(t,\,x\,(t)) - f\,(t,\,0),\,x\,(t) >,$$

according to Lemma 2.3 and (3.5), (3,6) we have

$$< f\,(t,\,x\,(t)),\,x\,(t) > \leqslant \gamma\,(t), \quad \text{for} \quad t \in [t_0\,,\,T],$$

$$< f\,(t,\,x\,(t)),\,x\,(t) > \leqslant -\,\beta\,(t), \quad \text{for} \quad t \in [T,\,\infty),$$

and by (3.4)

$$< g\,(t,\,x\,(t)),\,x\,(t) > \leqslant k\,(t)\,\|\,x\,(t)\,\|^2.$$

We take $\alpha\,(t) = \gamma\,(t)$ for $t \in [t_0\,,\,T)$ and $\alpha\,(t) = -\,\beta\,(t)$ for $t \in [T,\,\infty)$. Then (3.8) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\,x\,(t)\,\|^2 \leqslant 2\,(\alpha\,(t) + k\,(t))\,\|\,x\,(t)\,\|^2, \quad t \in [t_0\,,\,\infty),$$

and by integration

$$\|\,x\,(t)\,\| \leqslant \|\,x_0\,\| \exp \int_{t_0}^t (\alpha\,(s) + k(s))\,\mathrm{d}s\,, \quad t \in [t_0\,,\,\infty),$$

or still

$$\|\,x\,(t)\| \leqslant \|\,x_0\,\| \exp\left(\int_{t_0}^T (\gamma\,(s) + k\,(s))\,\mathrm{d}s\right) \exp\left(-\int_T^t (\beta\,(s) - k(s))\,\mathrm{d}s\right)$$

for $t \in [T,\,\infty)$.

Since $\gamma\,(t) + k\,(t)$ is bounded on $[t_0,\,T]$ we have $\int_{t_0}^T (\gamma\,(s) + k\,(s))\,\mathrm{d}s = M$ and therefore

$$\|\,x\,(t)\,\| \leqslant M \exp\left(-\int_T^t (\beta(s) - k\,(s))\,\mathrm{d}s\right), \quad t \in [T,\infty). \tag{3.9}$$

From this relation it follows the conclusion of the theorem.

Remark. The inequality (3.6) implies that the matrix $-\,J_s\,(t,\,x)$ is positive definite (or semi-definite) on $[T,\,\infty)$ with its smallest eigenvalue $\beta\,(t)$.

C o r o l l a r y. *Assume that assumption* (A) *is satisfied and* (3.5) *holds for* $t \in [t_0, \infty)$, $x \in X$. *If* $k(t) \leqslant 0$ *on* $[T, \infty)$ *and* $\gamma(t) \leqslant -k(t)$ *for* $t \in [T, \infty)$, *then the null solution is stable. If, in addition,* $\lim\limits_{t\to\infty} \int_T^t (\beta(s) + k(s))\, ds = -\infty$ *it is asymptotically stable.*

T h e o r e m 3.3. *Assume that the assumption* (A) *holds except with* (3.4) *replaced by*

$$< g(t, x), Vx > \leqslant k(t) \| x \|^2, \quad x \in X, \quad t \in [t_0, \infty), \tag{3.10}$$

where V *is a positive definite matrix. If the largest eigenvalue* $\Lambda(Vf'_x(t, x))$ *is such that*

$$\Lambda(Vf'_x(t, x)) \leqslant -\beta, \quad x \in X, \quad t \in [t_0, \infty) \tag{3.11}$$

where $\beta \geqslant 0$ and $\sup\limits_{t\in[T,\infty)} k(t) < \beta$ for some T, then the null solution is asymptotically stable.

P r o o f. Let $x = x(t)$ be any solution of (1.2) with the initial condition $x(t_0) = x_0$. Let $\| x(t) \|_v^2 = < x(t), Vx(t) >$ be. By Lemma 2.2

$$\frac{d}{dt} \| x(t) \|_v^2 = 2 < f(t, x(t)), Vx(t) > + 2 < g(t, x(t)), Vx(t) >.$$

On the other hand by Lemma 2.3 and (3.1) we have

$$2 < f(t, x(t)), Vx(t) > = 2 < Vf(t, x(t)), x(t) > =$$
$$= 2 < Vf(t, x(t)) - Vf(t, 0),$$
$$x(t) - 0 > \leqslant \Lambda(Vf'_x(t, x(t))) \| x(t) \|^2 \leqslant -\beta \| x(t) \|^2.$$

From here and taking into account (3.10) we get

$$\frac{d}{dt} \| x(t) \|_v^2 \leqslant -2(\beta - k(t)) \| x(t) \|^2, \quad t \in [t_0, \infty). \tag{3.12}$$

Since V is positive definite there exist two positive constants δ, $\bar{\delta}$ such that

$$\bar{\delta} \| x \|^2 \leqslant \| x \|_v^2 = < x, Vx > \leqslant \delta \| x \|^2, \tag{3.13}$$

and by hypothesis for some T $k = \sup\limits_{t\in[T,\infty)} k(t) < \beta$ it follows

$$\frac{d}{dt} \| x(t) \|_v^2 \leqslant -2(\beta - k) \| x(t) \|^2 \leqslant -2\lambda_1 \| x(t) \|_v^2, \quad t \in [T, \infty)$$

where $\lambda_1 = (\beta - k)/\delta > 0$. Integrating this inequality leads to

$$\| x(t) \|_v \leqslant \| x(T) \|_v \exp(-\lambda_1 (t - T)), \quad t \in [T, \infty). \tag{3.14}$$

On the other hand, for $t \in [t_0, T]$ from (3.12) it follows

$$(3.15) \qquad \| x(t) \|_v \leqslant \| x_0 \|_v \exp\left(\frac{1}{\delta} \int_{t_0}^{t} (k(s) - \beta)\, ds \right) \leqslant M_1 \| x_0 \|_v$$

where $M_1 = \sup_{t \in [t_0, T]} \left(\exp \frac{1}{\delta} \int_{t_0}^{t} (k(s) - \beta) \right) ds$.

From (3.13), (3.14) and (3.15) we get

$$\| x(t) \| \leqslant M \| x_0 \|, \quad t \in [t_0, T]$$

and

$$\| x(t) \| \leqslant M \exp(-\lambda_1 (t - T)) \| x_0 \|, \quad t \in [T, \infty)$$

whith a different constant M. From these two inequalities it follows that the null solution is asymptotically stable.

Corollary. *Under the same assumptions as in Theorem 3.3 except with* (3.10) *replaced by*

$$(3.16) \qquad \| g(t, x) \| \leqslant \varphi(t) \| x \|, \quad x \in X, \quad t \in [t_0, \infty)$$

where $\varphi(t)$ *is a continuous function on* $[t_0, \infty)$, *if for some* $T \geqslant t_0$ $\sup \varphi(t) < \beta/\delta$, *then the null solution is asymptotically stable.*

Indeed, by (3.13) and (3.16) we have

$$< g(t, x), Vx > = < g(t, x), x >_v \leqslant \| g(t, x) \|_v \| x \|_v \leqslant$$
$$\leqslant \delta \| x \| \| g(t, x) \| \leqslant \delta \varphi(t) \| x \|^2,$$

that is (3.10) with $k(t) = \delta \varphi(t)$.

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Polytechnic Institute, Jassy,
Department of Mathematics

REFERENCES

1. Demidoviĉ B. P., *On some Non-linear Systems with Levinson's D-property.* Proc. International Symp. Non-linear Vibrations, Kiev, 2 (1961), pp. 156–160.
2. Demidoviĉ B. P., *On Dissipativity of a Certain Non-linear System of Differential Equations* (I). V.M.U., 19–27 (1961).
3. Kato T., *Non-linear Semigroup and Evolution Equations.* J. Math. Soc. Jap., 19, 508–520 (1967).
4. Pao C. V., *On Stability of Non-linear Differential Systems.* Int. J. Non-linear Mech., 8, 219–238 (1973).

ASUPRA STABILITĂŢII SISTEMELOR DIFERENŢIALE NELINIARE

(Rezumat)

Se extind unele rezultate ale lui C. V. Pao la sistemele neliniare de forma (1.2) utilizînd o metodă a lui B. P. Demidovici privind construirea de produse scalare echivalente.

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ON A SYSTEM OF EVOLUTION NONLINEAR EQUATIONS IN A BANACH SPACE

BY

N. GRĂDINARU

Let A_1 and A_2 be two nonlinear operators with the domain and the range in a Banach space Y. Denote $E_2^+ = \{x = (x_1, x_2) \neq \theta = (0, 0), x_1 \geqslant 0, x_2 \geqslant 0\}$, $\Delta = [0, a_1] \times [0, a_2]$, where a_1, a_2 are arbitrary numbers.

Consider the problem [1] :

$$\begin{cases} u_{x_1} = A_1(u), \\ u_{x_2} = A_2(u), \end{cases} \quad x \in E_2^+, \tag{1}$$

$$u(\theta) = u_0 . \tag{2}$$

As in scalar case [2], we prove the following

Theorem. *Suppose $A_1(u)$, $A_2(u) : Y \to Y$ to be nonlinear, Fréchet differentiable operators, such that :*

(i) $\dfrac{\partial A_1(u)}{\partial u}, \dfrac{\partial A_2(u)}{\partial u}$ *are bounded linear operators on Y,*

(ii) $\left\| \dfrac{\partial A_1}{\partial u} \right\|, \left\| \dfrac{\partial A_2}{\partial u} \right\| \leqslant M$ *(M non depending on u),*

(iii) $$\frac{\partial A_1}{\partial u} A_2 = \frac{\partial A_2}{\partial u} A_1 .$$

Then the problem (1)–(2) *has a unique solution in $C^1(\Delta, Y)$.*

Proof. Consider the integral equation

$$u(x_1, x_2) = u_0 + \int_0^{x_1} A_1(u(s_1, 0))\, ds_1 + \int_0^{x_2} A_2(u(x_1, s_2))\, ds_2 \tag{3}$$

and the metric complete space $Z = C(\Delta, Y)$ metrized by $\rho(u_1, u_2) = \sup_{x \in \Delta} \| u_1 - u_2 \| \exp(-\tau(x_1 + x_2))$, $\tau > 0$. The operator K defined on Z by

$$Ku = u_0 + \int_0^{x_1} A_1(u(s_1, 0))\, ds_1 + \int_0^{x_2} A_2(u(x_1, s_2))\, ds_2$$

applies Z on Z and for $L/\tau < 1$, it is a strictly-contraction, where L is the Lipschitz constant of A_1 and A_2.

Indeed,

$$\| Ku_1 - Ku_2 \| \leqslant \int_0^{x_1} \| A_1(u_1(s_1, 0)) - A_1(u_2(s_1, 0)) \|\, ds_1 +$$

$$+ \int_0^{x_2} \| A_2(u_1(x_1, s_2)) - A_2(u_2(x_1, s_2)) \|\, ds_2 \leqslant$$

$$\leqslant L \int_0^{x_1} \| u_1(s_1, 0) - u_2(s_1, 0) \| \exp(-\tau s_1) \exp(\tau s_1)\, ds_1 +$$

$$+ L \int_0^{x_2} \| u_1(x_1, s_2) - u_2(x_1, s_2) \| \exp(-\tau(x_1 + s_2)) \exp(\tau(x_1 + s_2))\, ds_2 \leqslant$$

$$\leqslant \rho(u_1, u_2) \frac{L}{\tau} \exp(\tau(x_1 + x_2)).$$

Therefore

$$\rho(Ku_1, Ku_2) \leqslant \frac{L}{\tau} \rho(u_1, u_2).$$

Using the fixed point theorem of Banach, there exists a unique solution of the equation (3).

Let now prove the equivalence of the problem (1)—(2) with the equation (3). Being a solution of the problem (1)—(2) the function u satisfies the equation (3):

$$u(x_1, x_2) = u_0 + \int_0^{x_1} \frac{\partial u(s_1, 0)}{\partial s_1}\, ds_1 + \int_0^{x_2} \frac{\partial u(x_1, s_2)}{\partial s_2}\, ds_2 .$$

Conversely, if u is a solution of the equation (3), then

$$\frac{u(x_1, x_2 + h) - u(x_1, x_2)}{h} = \frac{1}{h} \int_{x_2}^{x_2+h} A_2(u(x_1, s_2))\, ds_2 \xrightarrow[h \to 0]{} A_2(u(x_1, x_2))$$

and therefore there exists $\partial u / \partial x_2$ and the second equation of (1) is satisfied. Because of the Fréchet differentiability of A_1, A_2, we obtain:

$$u(x_1+h, x_2) - u(x_1, x_2) = \int_{x_1}^{x_1+h} A_1(u(s_1, 0))\, ds_1 +$$

$$+ \int_0^{x_2} \left[\frac{\partial A_2(u(x_1, s_2))}{\partial u} \eta(x_1, s_2) + \omega(u(x_1, s_2), \eta) \right] ds_2$$

where

$$\eta(x_1, s_2) = u(x_1+h, s_2) - u(x_1, s_2), \quad \omega(u(x_1, s_2), \eta) / \| \eta \| \to \theta_y, \ \eta \to \theta_y.$$

We can write $\omega = \alpha(u, \eta) \| \eta \|$, where $\| \alpha \| \to 0$ for $\| \eta \| \to 0$. Therefore

$$\left\| \frac{\eta(x_1, x_2)}{h} \right\| \leqslant \| A_1(u(s_1', 0)) \| + \int_0^{x_2} (M + \| \alpha \|) \left\| \frac{\eta(x_1, s_2)}{h} \right\| ds_2,$$

where $s_1' \in (x_1, x_1+h)$ and for any $\varepsilon_1 > 0$, there exists $h_0(\varepsilon_1)$ such that

$$\| \alpha \| < \varepsilon_1 \quad \text{for} \quad h \leqslant h_0(\varepsilon_1).$$

Remember the lemma of Gronwall: if $z(t)$ is a real continuous function for $0 \leqslant t \leqslant a$ and if $0 \leqslant z(t) \leqslant \int_0^t [M_0 z(\sigma) + A]\, d\sigma + B$, with constants M_0, A, B, and $M_0 > 0$, then $0 \leqslant z(t) \leqslant (Aa + B) \exp(M_0 a)$, $0 < t < a$. Taking in this lemma $z = \| \eta/h \|$ we have: $\| \eta/h \| \leqslant M_1 \exp(M_0 + \varepsilon_1) a_2 = P = \text{const}$. Accordingly (iii), we can write:

$$\frac{\eta(x_1, x_2)}{h} = \frac{1}{h} \int_{x_1}^{x_1+h} A_1(u(s_1, 0))\, ds_1 + \int_0^{x_2} \left[\frac{\partial A_2}{\partial u} \frac{\eta(x_1, s_2)}{h} + \right.$$

$$\left. + \alpha(u, \eta) \left\| \frac{\eta(x_1, s_2)}{h} \right\| - \frac{\partial A_2}{\partial u} A_1(u(x_1, s_2)) + \frac{\partial A_1(u(x_1, s_2))}{\partial u} \frac{\partial u}{\partial s_2} \right] ds_2 =$$

$$= A_1(u(s_1', 0)) + \int_0^{x_2} \left[\frac{\partial A_2}{\partial u} \frac{\eta(x_1, s_2)}{h} + \alpha(u, \eta) \left\| \frac{\eta(x_1, s_2)}{h} - \right. \right.$$

$$\left. \left. - \frac{\partial A_2}{\partial u} A_1(u(x_1, s_2)) \right] ds_2 + A_1(u(x_1, x_2)) - A_1(u(x_1, 0)).$$

By continuity of $u(x_1, x_2)$ and $A_1(u)$, it follows that for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that for $h < \delta(\varepsilon)$ we have

$$\| A_1(u(s_1', 0)) - A_1(u(x_1, 0)) \| < \varepsilon, \quad s_1' \in (x_1, x_1+h), \quad \| \alpha(u, \eta) \| \leqslant \varepsilon.$$

Using (ii) we obtain

$$\left\| \frac{\eta(x_1, x_2)}{h} - A_1(u(x_1, x_2)) \right\| \leqslant \varepsilon +$$

$$+ \int_0^{x_2} \left[M \left\| \frac{\eta(x_1, s_2)}{h} - A_1(u(x_1, s_2)) \right\| + \varepsilon P \right] ds_2 .$$

Using again the lemma of Gronwall for

$$z = \left\| \frac{\eta(x_1, x_2)}{h} - A_1(u(x_1, x_2)) \right\|$$

we obtain

$$\left\| \frac{u(x_1 + h, x_2) - u(x_1, x_2)}{h} - A_1(u(x_1, x_2)) \right\| \leqslant (\varepsilon P a_2 + \varepsilon) \exp(M a_2)$$

and therefore there exists $\partial u(x_1, x_2) / \partial x_1 \in Z$ and the function u verifies also the first equation of (1). We have also, $u(0, 0) = u_0$.

The theorem is completely proved.

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Polytechnic Institute, Jassy,
Department of Mathematics

REFERENCES

1. Grădinaru N., *A Difference Scheme for a System of Nonlinear Equation of Evolution in Banach Spaces.* Bul. Inst. polit. Iaşi, **XXII (XXVI)**, *3–4*, s. I, 73–74 (1976).
2. Haimovici A., *Ecuaţii diferenţiale şi ecuaţii integrale.* EDP, Buc., 1965.

ASUPRA UNUI SISTEM DE ECUAŢII NELINIARE DE EVOLUŢIE ÎN SPAŢII BANACH

(Rezumat)

Se demonstrează o teoremă de existenţă a soluţiei problemei Cauchy (1)–(2), din [1] ca în [2, cap. XXVI], pentru sisteme de ecuaţii de evoluţie neliniare în spaţii Banach.

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MATEMATICĂ.
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ASUPRA SPAŢIILOR CU CONEXIUNE AFINĂ NESIMETRICĂ

DE

I. GHINEA

1. În această lucrare ne propunem să stabilim unele relaţii pentru spaţii cu conexiune afină nesimetrică, cu ajutorul conexiunii simetrice asociate şi al tensorului simetric de curbură introdus de Gh. Vrănceanu [1].

După cum se ştie [1], unui spaţiu A_n cu conexiune afină Γ nesimetrică, de coeficienţi Γ^i_{jk}, i se poate asocia o conexiune simetrică S, ai cărui coeficienţi S^i_{jk} sînt definiţi prin relaţiile

$$S^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} + \Gamma^i_{kj}) = \Gamma^i_{(jk)}, \tag{1}$$

şi un tensor de torsiune ale cărui componente T^i_{jk} sînt [2] [1])

$$T^i_{jk} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}) = \Gamma^i_{[jk]}. \tag{2}$$

Din relaţiile (1) şi (2) rezultă

$$\Gamma^i_{jk} = S^i_{jk} + T^i_{jk}. \tag{3}$$

Dacă ţinem seama de (3) în expresia tensorului de curbură

$$\Gamma^i_{jkl} = \partial_l \Gamma^i_{jk} - \partial_k \Gamma^i_{jl} + \Gamma^i_{sk}\Gamma^s_{jl} - \Gamma^i_{sl}\Gamma^s_{jk} \tag{4}$$

al spaţiului A_n, obţinem [1]

$$\Gamma^i_{jkl} = S^i_{jkl} + T^i_{jkl}, \tag{5}$$

unde

$$S^i_{jkl} = \partial_l S^i_{jk} - \partial_k S^i_{jl} + S^i_{sk} S^s_{jl} - S^i_{sl} S^s_{jk} \tag{6}$$

este tensorul de curbură corespunzător conexiunii simetrice S, care este antisimetric în indicii k, l, iar

$$T^i_{jkl} = \partial_l T^i_{jk} - \partial_k T^i_{jl} + T^i_{sk} T^s_{jl} - T^i_{sl} T^s_{jk} + T^i_{sk} S^s_{jl} - T^i_{sl} S^s_{jk} + T^s_{jl} S^i_{sk} - T^s_{jk} S^i_{sl} \tag{7}$$

sînt componentele unui tensor de asemenea antisimetric în indicii k, l.

---

[1]) În [1] Gh. Vrănceanu defineşte $T^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$. Am preferat relaţia (2) folosită de J. A. Schouten ([2], p. 126), deoarece conduce la relaţii mai simple în cele ce urmează.

2. În cele ce urmează se stabilesc cîteva relaţii între tensorul T^i_{jkl}, tensorul de curbură Γ^i_{jkl} şi tensorul de torsiune T^i_{jk}, cu ajutorul conexiunii simetrice S.

Fie

$$T^i_{jk/l} = \partial_l T^i_{jk} - T^s_{jk} \Gamma^i_{sl} + T^i_{sk} \Gamma^s_{jl} + T^i_{js} \Gamma^s_{kl} \tag{8}$$

derivata covariantă a tensorului T^i_{jk} în raport cu conexiunea Γ şi

$$T^i_{jk,l} = \partial_l T^i_{jk} - T^s_{jk} S^i_{sl} + T^i_{sk} S^s_{jl} + T^i_{js} S^s_{kl} \tag{9}$$

derivata covariantă a lui T^i_{jk} în raport cu conexiunea simetrică S.

Dacă în (8) ţinem seama de (3) şi (9) rezultă

$$T^i_{jk/l} = T^i_{jk,l} - T^s_{jk} T^i_{sl} + T^i_{sk} T^s_{jl} + T^i_{js} T^s_{kl} . \tag{10}$$

Relaţia (10) exprimă legătura între derivatele covariante ale tensorului T^i_{jk} în raport cu cele două conexiuni Γ şi S.

Permutînd circular indicii j, k, l şi sumînd, se obţine

$$T^i_{jk/l}+T^i_{kl/j}+T^i_{lj/k}=T^i_{jk,l}+T^i_{kl,j}+T^i_{lj,k}-3(T^i_{sj} T^s_{kl}+T^i_{sk} T^s_{lj}+T^i_{sl} T^s_{jk}). \tag{11}$$

Dacă se ţine seama de (9), expresia lui T^i_{jkl} dată de (7) devine

$$T^i_{jkl} = T^i_{jk,l} - T^i_{jl,k} + T^i_{sk} T^i_{jl} - T^i_{sl} T^s_{jk} . \tag{12}$$

Prin permutarea indicilor de covarianţă şi sumare, din (12) rezultă

$$T^i_{jkl}+T^i_{klj}+T^i_{ljk}=2(T^i_{jk,l}+T^i_{kl,j}+T^i_{lj,k})-2(T^i_{sj} T^s_{kl}+T^i_{sk} T^s_{lj}+T^i_{sl} T^s_{jk}). \tag{13}$$

Formulele (12) şi (13) dau relaţii între tensorul T^i_{jkl} şi tensorul de torsiune, independente de tensorul de curbură al conexiunii simetrice.

Dacă în (5) se permută indicii j, k, l şi se sumează, ţinînd seama că conexiunea S este simetrică, se obţine

$$\Gamma^i_{jkl} + \Gamma^i_{klj} + \Gamma^i_{ljk} = T^i_{jkl} + T^i_{klj} + T^i_{ljk} , \tag{14}$$

care oferă o relaţie între tensorul de curbură şi tensorul T^i_{jkl} independentă de tensorul S^i_{jkl}.

Din (13) şi (14) rezultă formulele

$$\Gamma^i_{jkl}+\Gamma^i_{klj}+\Gamma^i_{ljk}=2(T^i_{jk,l}+T^i_{kl,j}+T^i_{lj,k})-2(T^i_{sj} T^s_{kl}+T^i_{sk} T^s_{lj}+T^i_{sl} T^s_{jk}), \tag{15}$$

care stabilesc o relaţie între tensorul de curbură şi tensorul de torsiune.

Fie

$$\Gamma^i_{jkl/r} = \partial_r \Gamma^i_{jkl} - \Gamma^s_{jkl} \Gamma^i_{sr} + \Gamma^i_{skl} \Gamma^s_{jr} + \Gamma^i_{jsl} \Gamma^s_{kr} + \Gamma^i_{jks} \Gamma^s_{lr} \tag{16}$$

şi

$$\Gamma^i_{jkl,h} = \partial_r \Gamma^i_{jkl} - \Gamma^s_{jkl} S^i_{sr} + \Gamma^i_{skl} S^s_{jr} + \Gamma^i_{jsl} S^s_{kr} + \Gamma^i_{jks} S^s_{lr} \tag{17}$$

derivatele covariante ale lui Γ^i_{jkl} în raport cu conexiunea Γ, respectiv S.

Dacă în (16) ţinem seama de (3) rezultă că

$$\Gamma^i_{jkl/r} = \Gamma^i_{jkl,r} - \Gamma^s_{jkl} T^i_{sr} + \Gamma^i_{skl} T^s_{jr} + \Gamma^i_{jsl} T^s_{kr} + \Gamma^i_{jks} T^s_{lr} , \tag{18}$$

care exprimă legătura între derivatele covariante ale tensorului în raport cu cele două conexiuni, Γ şi S.

Prin permutarea circulară a indicilor k, l, r şi sumare se obţin relaţiile

$$\Gamma^i_{j[kl/r]} = \Gamma^i_{j[kl,r]} + \Gamma^s_{j[kl.} T^i_{r]s} - \Gamma^i_{s[kl.} T_{r]j} - 2\,\Gamma^i_{js[k.} T^s_{lr]}\,. \tag{19}$$

Să considerăm acum derivatele covariante ale relaţiei (5) în raport cu conexiunea simetrică S :

$$\Gamma^i_{jkl,r} = S^i_{jkl,r} + T^i_{jkl,r}\,. \tag{20}$$

Dacă se face o permutare circulară a indicilor k, l, r şi se sumează ţinînd seama că derivatele covariante ale tensorului de curbură S^i_{jkl} satisfac identităţile lui Bianchi, se obţine

$$B^i_{jklr} \equiv \Gamma^i_{jkl,r} + \Gamma^i_{jlr,k} + \Gamma^i_{jrk,l} - (T^i_{jkl,r} + T^i_{jlr,k} + T^i_{jrk,l}) = 0. \tag{21}$$

Analog, din (5), prin derivare covariantă în raport cu conexiunea simetrică S şi ţinînd seama de identităţile lui Veblen pentru S^i_{jkl}, rezultă

$$V^i_{jklr} \equiv \Gamma^i_{jkl,r} + \Gamma^i_{ljr,h} + \Gamma^i_{rlk,j} + \Gamma^i_{krj,l} - (T^i_{jkl,r} + T^i_{ljr,k} + T^i_{rlk,j} + T^i_{krj,l}) = 0. \tag{22}$$

Utilizînd derivatele covariante ale formulelor (14) în raport cu conexiunea simetrică S, se verifică următoarele relaţii :

$$B^i_{jklr} + B^i_{lrjk} = V^i_{kjlr}\,, \qquad 2B^i_{jklr} = V^i_{jklr} + V^i_{jlrk} + V^i_{jrkl}\,. \tag{23}$$

Rezultatul (16) arată că identităţile (21) şi (22) sînt echivalente. Această echivalenţă este analogă cu echivalenţa identităţilor lui Bianchi şi Veblen, valabilă pentru tensorii de curbură în cazul spaţiilor A_n fără torsiune, echivalenţă stabilită în [3].

3. Să asociem acum, după Gh. V r ă n c e a n u [1], spaţiului A_n tensorul

$$G^i_{jkl} = \Gamma^i_{jkl} + \Gamma^i_{kjl} = \Gamma^i_{jkl} - \Gamma^i_{klj}\,, \tag{24}$$

numit tensor simetric de curbură, deoarece este simetric în indicii j, k.

Dacă în relaţiile (24) se ţine seama de (4), (3) şi (10), se obţine

$$\begin{aligned} G^i_{jkl} = {} & 2\partial_l S^i_{jk} - \partial_k S^i_{jl} - \partial_j S^i_{kl} - (T^i_{jl,k} + T^i_{kl,j}) + \\ & + (T^i_{sk} T^s_{jl} + T^i_{sj} T^s_{kl}) + S^i_{sk} S^s_{jl} + S^i_{sj} S^s_{kl} - 2S^i_{sl} S^s_{jk}\,. \end{aligned} \tag{25}$$

Folosind pe (6) în (25) rezultă

$$G^i_{jkl} = S^i_{jkl} + S^i_{kjl} - (T^i_{jl,k} + T^i_{kl,j}) + (T^i_{sk} T^s_{jl} + T^i_{sj} T^s_{kl})\,. \tag{26}$$

Formulele (26) dau tensorul simetric de curbură G^i_{jkl} în funcţie de tensorul de curbură S^i_{jkl} al conexiunii simetrice S, de tensorul de torsiune T^i_{jk} şi de derivatele covariante ale acestuia în raport cu conexiunea S.

Notînd

$$T^i_{sj} T^s_{kl} = T^s_{kl} T^i_{sj} = U^i_{jkl} \tag{27}$$

rezultă că U^i_{jkl} este un tensor antisimetric în ultimii doi indici de covarianţă, iar (26) devine

$$G^i_{jkl} = S^i_{jkl} + S^i_{kjl} - (T^i_{jl,k} + T^i_{kl,j}) + (U^i_{jkl} + U^i_{kjl}). \tag{26'}$$

În cazul particular cînd conexiunea Γ este simetrică, din (26) rezultă

$$G^i_{jkl} = S^i_{jkl} + S^i_{kjl}\,.$$

Dacă se permută acum în formula (26), circular, indicii de covarianţă şi se sumează, rezultă

$$G^i_{jkl} + G^i_{klj} + G^i_{ljk} = 0, \tag{28}$$

fapt stabilit în [1], dar folosind relaţia de definiţie (24) a lui G^i_{jkl}.

Ţinînd seama de (15), relaţia (24) conduce la

$$G^i_{jkl} = \bar{S}^i_{jkl} + T^i_{jkl}\,, \tag{29}$$

în care

$$\bar{S}^i_{jkl} = S^i_{jkl} + S^i_{kjl} = S^i_{jkl} - S^i_{klj} \tag{30}$$

este tensorul simetric de curbură corespunzător conexiunii simetrice asociate S, iar

$$\bar{T}^i_{jkl} = T^i_{jkl} + T^i_{kjl} = T^i_{jkl} - T^i_{klj} \tag{31}$$

este un tensor de ordinul patru, cu valenţele indicate de poziţia indicilor, simetric în raport cu primii doi indici de covarianţă.

Din proprietăţile tensorilor S^i_{jkl} şi T^i_{jkl} rezultă cele ale tensorilor $\bar{S}^i_{jkl}$ şi $\bar{T}^i_{jkl}$. Astfel, relaţiile

$$\bar{S}^i_{jkl} + \bar{S}^i_{klj} + \bar{S}^i_{ljk} = 0 \quad \text{şi} \quad \bar{T}^i_{jkl} + \bar{T}^i_{klj} + \bar{T}^i_{ljk} = 0 \tag{32}$$

rezultă imediat. De asemenea, dacă în (31) se ţine seama de (12), se obţine

$$T^i_{jkl} = -(T^i_{jl,k} + T^i_{kl,j}) + T^i_{sk}\,T^s_{jl} + T^i_{sj}\,T^s_{kl}\,. \tag{33}$$

Dacă în (29) se ţine seama de (30) şi (33), se regăseşte (26).

Fie acum derivată relaţia (24), covariant în raport cu conexiunea Γ. Avînd în vedere rezultatele (18) şi (24) se obţine

$$G^i_{jkl/r} = G^i_{jkl,r} - G^s_{jkl}\,T^i_{sr} + G^i_{skl}\,T^i_{jr} + G^i_{jkl}\,T^s_{kr} + G^i_{jks}\,T^s_{lr}\,, \tag{34}$$

care dă legătura între derivatele covariante ale tensorului simetric G^i_{jkl} în raport cu conexiunile Γ şi S. Se observă analogia cu (18).

Prin adunarea lui (14), respectiv (15), cu

$$\Gamma^i_{jkl} - \Gamma^i_{klj} = G^i_{jkl} \quad \text{şi} \quad -\Gamma^i_{ljk} + \Gamma^i_{jkl} = -G^i_{jlk}\,,$$

care rezultă din (24), se obţine

$$3\Gamma^i_{jkl} = G^i_{jkl} - G^i_{jlk} + (T^i_{jkl} + T^i_{klj} + T^i_{ljk}) \tag{35}$$

respectiv

$$(36)\quad 3\Gamma^i_{jkl}=G^i_{jkl}-G^i_{jlk}+2(T^i_{jk,l}+T^i_{kl,j}+T^i_{lj,k})-2(T^i_{sj}T^s_{kl}+T^i_{sk}T^s_{lj}+T^i_{sl}T^s_{jk}).$$

Formulele (35) şi (36) denotă că curbura spaţiului A_n poate fi definită cu ajutorul tensorilor G^i_{jkl} şi T^i_{jkl} sau al tensorilor G^i_{jkl} şi T^i_{jk}.

Dacă în relaţia (21), amplificată cu 3, se înlocuiesc derivatele covariante conţinute prin expresiile corespunzătoare obţinute din (35), rezultă

$$(37)\quad \mathcal{B}^i_{jklr}\equiv G^i_{i[kl,r]}-G^i_{j[lk,r]}+(T^i_{[jkl],r}+T^i_{[jlr],k}+T^i_{[jrk],l})-3T^i_{j[kl,r]}=0.$$

Aceste formule dau relaţiile dintre derivatele covariante în raport cu conexiunea simetrică ale tensorilor G^i_{jkl} şi T^i_{jkl}.

În cazul particular cînd conexiunea Γ este simetrică, se obţin relaţiile

$$(38)\quad G^i_{j[kl,r]}-G^i_{j[lk,r]}=0\,,$$

adică identităţile lui Bianchi pentru tensorul de curbură G^i_{jkl} obţinute în [4] în ipoteza că conexiunea Γ este simetrică.

În mod analog, dacă în (22) înmulţită cu 3 se înlocuiesc derivatele covariante prin expresiile corespunzătoare obţinute din (35), rezultă

$$(39)\quad \begin{aligned}\mathcal{V}^i_{jklr}\equiv{}& G^i_{jkl,r}+G^i_{ljr,k}+G^i_{rlk,j}+G^i_{krj,l}-G^i_{jlk,r}-G^i_{lrj,k}-G^i_{rkl,j}-G^i_{kjr,l}+\\ &+3[T^i_{[jkl],r}+T^i_{[ljr],k}+T^i_{[rlk],j}+T^i_{[krj],l}]-3(T^i_{jkl,r}+T^i_{ljr,k}+T^i_{rlk,j}+T^i_{krj,l})=0.\end{aligned}$$

În cazul particular cînd conexiunea Γ este simetrică se obţin identităţile lui Veblen pentru tensorul simetric de curbură:

$$(40)\quad G^i_{jkl,r}+G^i_{ljr,k}+G^i_{rlk,j}+G^i_{krj,l}-G^i_{jlk,r}-G^i_{lrj,k}-G^i_{rkl,j}-G^i_{kjr,l}=0,$$

stabilite în [4], în ipoteza că conexiunea este simetrică. Şi în acest caz, ţinînd seama de derivatele covariante ale lui (35), se verifică relaţiile

$$(41)\quad \mathcal{B}^i_{jklr}+\mathcal{B}^i_{lrjk}=\mathcal{V}^i_{kjlr}\,,\qquad 2\mathcal{B}^i_{jklr}=\mathcal{V}^j_{jklr}+\mathcal{V}^i_{jlrk}+\mathcal{V}^i_{jrkl}\,,$$

care arată că identităţile (37) şi (39) sînt echivalente.

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Institutul politehnic, Iaşi,
Catedra de matematică

BIBLIOGRAFIE

1. Vrănceanu Gh., *Leçons de géométrie différentielle.* V. I, Ed. de l'Académie, Bucarest, 1957.
2. Schouten J. A., *Ricci-Calculus.* Second ed., Springer Verlag, Berlin — Göttingen — Heidelberg, 1954.
3. Blum R., *Sur les identités de Bianchi et Veblen.* C.r. Acad. Sc. Paris, 224, 889 (1947).
4. Ispas C., *Identităţile lui Bianchi şi Veblen pentru tensorul simetric de curbură al unui spaţiu cu conexiune afină. Echivalenţa lor.* St. şi cerc. mat., Buc., 19, 461—467 (1967)

SUR LES ESPACES À CONNEXION AFFINE NONSYMÉTRIQUE

(Résumé)

On établi une série de relations concernant les espaces à connexion affine nonsymétrique, en utilisant une connexion symétrique associée et le tenseur symétrique de courbure introduit par Gh. Vrănceanu [1],

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SUR LES TENSEURS PROJECTIFS DANS LES ESPACES FINSLER (u, a)-RÉCURRENTS ET (v, b)-RÉCURRENTS

PAR

GH. ANDRICIOAEI

0. Cette note est une continuation de notre précédent travail [3]. Ici, nous allons étudier les (u, a)-récurrence et (v, b)-récurrence des tenseurs projectifs.

1. Les tenseurs projectifs de l'espace Finsler donnés par [1]

$$W^i_{jkh} = H^i_{jkh} + \frac{\delta^i_j}{n+1}(H_{kh} - H_{hk}) + \frac{\dot{x}^i}{n+1}\left(\frac{\partial H_{kh}}{\partial \dot{x}^j} - \frac{\partial H_{hk}}{\partial \dot{x}^j}\right) + \tag{1.1}$$
$$+ \frac{\delta^i_k}{n^2-1}\left(nH_{jh} + H_{hj} + \dot{x}^r \frac{\partial H_{hr}}{\partial \dot{x}^j}\right) - \frac{\delta^i_h}{n^2-1}\left(nH_{jk} + H_{kj} + \dot{x}^r \frac{\partial H_{kr}}{\partial \dot{x}^j}\right)$$

$$W^i_{jk} = H^i_{jk} + \frac{\dot{x}^i}{n+1}(H_{jk} - H_{kj}) + \tag{1.2}$$
$$+ \frac{\delta^i_j}{n^2-1}(nH_k + \dot{x}^r H_{kr}) - \frac{\delta^i_k}{n^2-1}(nH_j + \dot{x}^r H_{jr}),$$

$$W^i_k = H^i_k - H\delta^i_k - \frac{1}{n+1}\left(\frac{\partial H^j_k}{\partial \dot{x}^j} - \frac{\partial H}{\partial \dot{x}^k}\right)\dot{x}^j, \tag{1.3}$$

satisfont les relations

$$W^i_{jkh}\dot{x}^j = W^i_{kh}, \quad W^i_{jk}\dot{x}^j = W^i_k, \quad W^i_k\dot{x}^k = 0, \quad \frac{\partial W^i_k}{\partial \dot{x}^r}\dot{x}^k = -W^i_r. \tag{1.4}$$

Dans ce qui suit, nous utiliserons la formule de commutation

$$\left(\frac{\partial T^i_j}{\partial \dot{x}^k}\right)_{(h)} - \frac{\partial T^i_{j(h)}}{\partial \dot{x}^k} = T^i_r G^r_{jkh} - T^r_j G^r_{rkj}, \tag{1.5}$$

où

$$G^i_{hjk} = \frac{\partial G^i_{hj}}{\partial \dot{x}^k} = \frac{\partial G^i_{kj}}{\partial \dot{x}^h}.$$

2. Dans [3] nous avons introduit les notions d'espace Finsler (u, a)-récurrent et (v, b)-récurrent, par

$$H^i_{hjk(l)(m)} = u_m H^i_{hjk(l)} + a_{lm} H^i_{hjk} \tag{2.1 a}$$

et respectivement

$$H^i_{hjk(l)(m)} = v_l H^i_{hjk(m)} + b_{lm} H^i_{hjk}. \tag{2.1 b}$$

De ces relations il résulte

$$H^i_{jk(l)(m)} = u_m H^i_{jk(l)} + a_{lm} H^i_{jk}, \tag{2.2 a}$$

$$H^i_{jk(l)(m)} = v_l H^i_{jk(m)} + b_{lm} H^i_{jk} \tag{2.2 b}$$

et

$$H_{hj(l)(m)} = u_m H_{hj(l)} + a_{lm} H_{hj}, \tag{2.3 a}$$

$$H_{hj(l)(m)} = v_l H_{hj(m)} + b_{lm} H_{hj}. \tag{2.3 b}$$

En ce qui concerne les tenseurs projectifs W^i_{jk} et W^i_k, nous avons le

Théorème 2.1. *Dans les espaces Finsler (u, a)-récurrent et (v, b)-récurrent, les tenseurs projectifs W^i_{jk} et W^i_k satisfont les relations de récurrence*

$$\left\{\begin{array}{ll} \text{a)} & W^i_{jk(l)(m)} = u_m W^i_{jk(l)} + a_{lm} W^i_{jk}, \\ \text{b)} & W^i_{jk(l)(m)} = v_l W^i_{jk(m)} + b_{lm} W^i_{jk}; \end{array}\right. \tag{2.4}$$

$$\left\{\begin{array}{ll} \text{a)} & W^i_{k(l)(m)} = u_m W^i_{k(l)} + a_{lm} W^i_k, \\ \text{b)} & W^i_{k(l)(m)} = v_l W^i_{k(m)} + b_{lm} W^i_k. \end{array}\right. \tag{2.5}$$

En effet, en dérivant covariant (1.2) par rapport à x^l et x^m et en tenant compte de (2.2 a) et (2.2 b), on obtient les relations (2.4 a) et (2.4 b).

Puis, en multipliant (2.4 a) et (2.4 b) avec $\dot{x}^j$ et en utilisant (1.4), on obtient (2.5 a) et (2.5 b).

Pour la bi-récurrence du tenseur W^i_{jkh} nous avons le

Théorème 2.2. *Si dans les espaces Finsler (u, a)-récurrent et (v, b)-récurrent, les vecteurs u, v, et les tenseurs a et b sont indépendants des arguments directionels et si la relation*

$$G^i_{jkh} = 0 \tag{2.6}$$

est satisfaite, alors le tenseur W^i_{jkh} satisfait aussi les relations (2.1 a) *et* (2.1 b).

Démonstration. En dérivant covariant deux fois la relation (1.1), on a

$$\begin{aligned} W^i_{jkh(l)(m)} &= H^i_{jkh(l)(m)} + (H_{kh(l)(m)} - H_{hk(l)(m)}) + \\ &+ \frac{\dot{x}^i}{n+1}\left[\left(\frac{\partial H_{kh}}{\partial \dot{x}^j}\right)_{(l)(m)} - \left(\frac{\partial H_{hk}}{\partial \dot{x}^j}\right)_{(l)(m)}\right] + \\ &+ \frac{\delta^i_k}{n^2-1}\left(nH_{jh} + H_{hj} + \dot{x}^r \frac{\partial H_{hr}}{\partial \dot{x}^j}\right)_{(l)(m)} - \frac{\delta^i_h}{n^2-1}\left(nH_{jk} + H_{kj} + \dot{x}^r \frac{\partial H_{kr}}{\partial \dot{x}^j}\right)_{(l)(m)} \end{aligned} \tag{2.7}$$

D'après la formule de commutation (1.5) et de (2.6), nous avons

$$\left(\frac{\partial H_{kh}}{\partial \dot{x}^j}\right)_{(l)} = \frac{\partial H_{kh(l)}}{\partial \dot{x}^j} + H_{rh}\, G^r_{kjl} + H_{kr}\, G^r_{hjl} = \frac{\partial H_{kh(l)}}{\partial \dot{x}^j} \tag{2.8}$$

et

$$\left(\frac{\partial H_{kh}}{\partial \dot{x}^j}\right)_{(l)(m)} = \frac{\partial H_{kh(l)(m)}}{\partial \dot{x}^j}. \tag{2.9}$$

Si on tient compte de ces relations, de (2.3 a), (2.3 b) et du fait que u, v, a et b sont indépendants des arguments directionels, la relation (2.7) donne

$$W^i_{jkh(l)(m)} = u_m\, W^i_{jkh(l)} + a_{lm}\, W^i_{jkh} \tag{2.10}$$

et

$$W^i_{jkh(l)(m)} = v_l\, W^i_{jkh(m)} + b_{lm}\, W^i_{jkh}\,, \qquad \text{c.q.f.d.} \tag{2.11}$$

Remarque. Si dans un espace Finsler on a des relations (2.10) et (2.11), alors en multipliant successivement ces relations avec $\dot{x}^j$ et $\dot{x}^k$, on obtient les relations (2.4) et (2.5).

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Institut Polytechnique, Jassy
Chaire de Mathématiques

BIBLIOGRAPHIE

1. Rund H., *The Differential Geometry of Finsler Spaces.* Springer-Verlag, 1959.
2. Sinha B. B., Singh Ş. P., *Recurrent Finsler Spaces of Second Order* (II). Indian J. Pure Appl. Math., 4, *1*, 45–50 (1973)
3. Andricioaei Gh., *Sur les espaces Finsler bi-récurrents.* Bull. Inst. polit. Iaşi, XXII (XXVI), *3–4*, s. I (1976).

ASUPRA TENSORILOR PROIECTIVI ÎN SPAŢII FINSLER (u, a)-RECURENTE ŞI (v, b)-RECURENTE

(Rezumat)

Se studiază tensorii proiectivi ai spaţiilor Finsler (u, a)-recurente şi (v, b)-recurente introduse în lucrarea anterioară [3], arătîndu-se că tensorii W^i_{jk} şi W^i_k satisfac relaţiile de recurenţă (2.1 a) şi (2.1 b), iar tensorul W^i_{hjk} satisface relaţiile respective în condiţia (2.6).

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ECUAŢII DIFERENŢIALE CU SINGULARITĂŢI, ÎN SPAŢII BANACH

DE

I. ONCIULENCU şi VIOREL GHIUŞ

Ecuaţia pe care o vom studia în această notă reprezintă o generaizare în spaţii Banach a ecuaţiei diferenţiale de ordinul doi, de tip Fuchs, de la cazul scalar.

Fie ecuaţia :

$$y'' + \varphi_1(p_1(z), y') + \varphi_2(p_2(z), y) = 0, \tag{1}$$

în care

a) $\varphi_1 : E \times E \to E$ şi $\varphi_2 : E \times E \to E$

sînt forme biliniare continui, E fiind un spaţiu Banach oarecare peste C (cîmpul complex)

b)
$$p_1 : U \setminus \{0\} \to E, \quad p_1(z) = \frac{1}{z}\sum_{i=0}^{\infty} A_i z^i, \quad A_i \in E,$$
$$p_2 : U \setminus \{0\} \to E, \quad p_2(z) = \frac{1}{z^2}\sum_{i=0}^{\infty} B_i z^i, \quad B_i \in E,$$

iar $\sum_{i=0}^{\infty} A_i z^i$ şi $\sum_{i=0}^{\infty} B_i z^i$ sînt convergente în vecinătatea U a originii din C.

În anumite ipoteze despre φ_1 şi φ_2, vom arăta că ecuaţia (1) are soluţii de forma

$$y = z^r \sum_{i=0}^{\infty} C_i z^i, \quad C_0 \neq 0, \quad C_i \in E. \tag{2}$$

Pentru aceasta, derivăm formal relaţia (2), de două ori şi înlocuim apoi rezultatele în (1). Obţinem :

$$y' = z^{r-1}\sum_{i=0}^{\infty} C_i (r+i) z^i, \qquad y'' = z^{r-2}\sum_{i=0}^{\infty} C_i (r+i)(r+i-1) z^i$$

şi

$$\sum_{i=0}^{\infty} (r+i)(r+i-1) z^i + \varphi_1\left(\sum_{i=0}^{\infty} A_i z^i \sum_{i=0}^{\infty} C_i (r+i) z^i\right) +$$

$$+ \varphi_2\left(\sum_{i=0}^{\infty} B_i z^i, \sum_{i=0}^{\infty} C_i z^i\right) = 0 ;$$

grupînd convenabil se obţine

$$\sum_{n=0}^{\infty} [(r+n)(r+n-1) C_n + \sum_{m=0}^{n} (\varphi_1 (A_{n-m}, (r+m) C_m) +$$

$$+ \varphi_2 (B_{n-m}, C_m))] z^n = 0.$$

Se impun astfel condiţiile :

$$(3) \qquad \begin{aligned} &(r+n)(r+n-1) C_n + \sum_{m=0}^{n} (\varphi_1 (A_{n-m}, (r+m) C_m) + \\ &+ \varphi_2 (B_{n-m}, C_m)) = 0 \qquad (n = 0, 1, 2, \ldots). \end{aligned}$$

Condiţiile (3) le scriem în forma :

$$(3') \qquad \begin{aligned} &C_n (r+n)(r+n-1) + (r+n) \varphi_1 (A_0, C_n) + \varphi_2 (B_0, C_n) + \\ &+ \sum_{m=0}^{n-1} [\varphi_1 (A_{n-m}, (r+m) C_m) + \varphi_2 (B_{n-m}, C_m)] = 0. \end{aligned}$$

Făcînd în (3′) $n = 0$ obţinem ecuaţia :

$$(4) \qquad r(r-1) C_0 + r \varphi_1 (A_0, C_0) + \varphi_2 (B_0, C_0) = 0.$$

Ecuaţia (4) o numim ecuaţia caracteristică a ecuaţiei diferenţiale (1). Fie acum : $T : C \setminus \{0, 1\} \to L(E, E)$,

$$T(z) = -\frac{1}{z(z-1)} [z \varphi_1 (A_0, .) + \varphi_2 (B_0, .)].$$

Presupunem că operatorul T are următoarea proprietate :

(I_1) $\quad \exists r_0$, încît $T(r_0) = I \in L(E, E)$, unde $Ix = x, \quad \forall x \in E$.

Asta înseamnă că are loc egalitatea :

$$(4') \qquad r_0 (r_0 - 1) C + r_0 \varphi_1 (A_0, C) + \varphi_2 (B_0, C) = 0, \quad \forall C \in E.$$

Fie $C = C_0 \neq 0$, fixat pentru cele ce urmează. Atunci ecuaţiile (3) în care înlocuim pe r cu r_0, devin, ţinînd seama de (4′),

$$n(2r_0 + n - 1) C_n + n\varphi_1 (A_0, C_n) + \sum_{m=0}^{n-1} [\varphi_1 (A_{n-m}, (r+m) C_m) +$$

$$+ \varphi_2 (B_{n-m}, C_m)] = 0,$$

de unde

$$(3'')\quad C_n = -\frac{1}{2r_0+n-1}\varphi_1(A_0, C_n) - \frac{1}{n(2r_0+n-1)}\sum_{m=0}^{n-1}[\varphi_1(A_{n-m}, (r+m)C_m) + \varphi_2(B_{n-m}, C_m)].$$

Notînd :

$$b_{n-1} = -\frac{1}{n(2r_0+n-1)}\sum_{m=0}^{n-1}[\varphi_1(A_{n-m}, (r+m)C_m) + \varphi_2(B_{n-m}, C_m)]$$

şi cu T_n operatorul :

$$T_n : E \to E, \quad T_n X = -\frac{1}{2r_0+n-1}\varphi_1(A_0, x),$$

(3″) se scrie în forma :

$$(5)\qquad C_n = T_n C_n + b_{n-1}.$$

Avem :

$$\| T_n x - T_n y \| = \frac{1}{|2r_0+n-1|}\| \varphi_1(A_0, x-y) \| \leqslant$$

$$\leqslant \frac{\|\varphi_1\| \, \|A_0\|}{|2r_0+n-1|}\| x-y \|$$

şi evident, pentru

$$u \geqslant u_0, \quad \frac{\|\varphi_1\| \, \|A_0\|}{|2r_0+n-1|} \leqslant \alpha < 1.$$

Mai presupunem că φ_1, satisface condiţiile :

$$(\mathrm{I}_2)\qquad \frac{\|\varphi_1\| \, \|A_0\|}{|2r_0+n-1|} \leqslant \alpha < 1, \quad n = 1, 2, \ldots$$

adică operatorii $T_1, T_2, \ldots, T_n, \ldots$ sînt contractanţi.

Ipotezele (I_1) şi (I_2) asigură determinarea seriei formale

$$y = z^{r_0}\sum_{i=0}^{\infty} C_i z^i.$$

Vom demonstra în continuare convergenţa seriei formale găsită anterior.

Din (5) rezultă :

$$\| C_n \| \leqslant \| T_n \| \, \| C_n \| + \| b_{n-1} \|$$

şi cum

$$\| T_n \| \leqslant \frac{\|\varphi_1\| \, \|A_0\|}{|2r_0+n-1|} \leqslant \alpha < 1$$

obţinem că

$$\| C_n \| \leqslant \frac{1}{1-\alpha} \| b_{n-1} \| = \frac{1}{1-\alpha} \frac{1}{|2r_0+n-1|} \left\| \sum_{m=0}^{n-1} [\varphi_1(A_{n-m}, (r_0+m) C_m + \right.$$

$$\left. + \varphi_2(B_{n-m}, C_m)] \right\| \leqslant \frac{1}{1-\alpha} \frac{1}{n^2} \frac{n}{|2r_0+n-1|} \sum_{m=0}^{n-1} [\|\varphi_1\| \|A_{n-m}\| (|r_0|+m) +$$

$$+ \|\varphi_2\| \|B_{n-m}\|] \|C_m\| .$$

Însă

$$\frac{n}{|2r_0+n-1|} \leqslant k, \quad n = 1, 2, \ldots, \qquad \|A_n\| \leqslant \frac{M}{\rho^n}, \quad n = 1, 2 \ldots$$

$\|B_n\| \leqslant \dfrac{N}{\rho^n}$, ([2]), M, N, K, depinzînd de vecinătatea U.

Aceste ultime evaluări conduc la inegalităţile:

$$\|\varphi_1\| \|A_n\| \leqslant \frac{M_1}{\rho^n}, \quad n = 1, 2, \ldots \quad \text{şi} \quad \|\varphi_2\| \|B_n\| \leqslant \frac{N_1}{\rho^n}, \quad n = 1, 2 \ldots$$

şi de aici la

$$\text{(6)} \qquad \| C_n \| \leqslant \frac{k}{1-\alpha} \sum_{m=0}^{n-1} \frac{M_1(|r_0|+m)+N_1}{\rho^{n-m}}, \quad n = 1, 2, \ldots$$

Datorită inegalităţilor (6) rezultă că seria $\sum_{n=0}^{\infty} \| C_n \| z^n$ este convergentă pentru $z \in U' = \{z \mid z \in C, |z| \leqslant \rho\}$. Astfel, în ipotezele (I_1) şi (I_2), ecuaţia (1) admite soluţii

$$y = z^r \sum_{m=0}^{\infty} C_n z^n, \qquad z \in U' \setminus \{0\},$$

U' fiind o vecinătate a originii din C.

Exemplu. Fie $E = C^2$. Dacă $x = (x_1, x_2) \in E$, $y = (y_1, y_2) \in E$ atunci

$$\text{a)} \qquad \varphi_1(x, y) = (\lambda_1^1(x) y_1 + \lambda_2^1(x) y_2, \quad \lambda_1^2(x) y_1 + \lambda_2^2(x) y_2),$$

unde

$$\lambda_i^j(x) = \lambda_{i1}^j x_1 + \lambda_{i2}^j x_2, \quad i, j = 1, 2, \quad \lambda_{ij}^k \in C,$$

iar

$$\varphi_2(x, y) = (\gamma_1^1(x) y_1 + \gamma_2^1(x) y_2, \quad \gamma_1^2(x) y_1 + \gamma_2^2(x) y_2),$$

cu

$$\gamma_i^j(x) = \gamma_{i1}^j x_1 + \gamma_{i2}^j x_2, \quad i, j = 1, 2, \quad \gamma_{i,j}^k \in C.$$

$$\text{b)} \qquad p_1(z) = (p_1^1(z), p_2^1(z)), \quad z \in C,$$

cu

$$p_1^1(z) = \frac{1}{z}(A_0^1 + A_1^1 z + \ldots),$$

$$p_2^1(z) = \frac{1}{z}(A_0^2 + A_1^2 z + \ldots); \qquad A_0 = (A_0^1, A_0^2),$$

ar

$$p_2(z) = (p_1^1(z), p_2^2(z)),$$

cu

$$p_1^2(z) = \frac{1}{z^2}(B_0^1 + B_1^1 z + B_2^1 z^2 + \ldots),$$

$$p_2^2(z) = \frac{1}{z^2}(B_0^2 + B_1^2 z + B_2^2 z^2 + \ldots); \qquad B_0 = (B_0^1, B_0^2).$$

Ecuaţia caracteristică (4) se scrie acum în forma

$$r(r-1)(C_0^1, C_0^2) + r(\lambda_1^1(A_0)C_0^1 + \lambda_2^1(A_0)C_0^2, \lambda_1^2(A_0)C_0^1 + \lambda_2^2(A_0)C_0^2) +$$
$$+ (\gamma_1^1(B_0)C_0^1 + \gamma_2^1(B_0)C_0^2, \gamma_1^2(B_0)C_0^1 + \gamma_2^2(B_0)C_0^2) = 0.$$

Condiţia (I_1) devine echivalentă cu sistemul:

$$(4') \quad \begin{cases} r(r-1) + r\lambda_1^1(A_0) + \gamma_1^1(B_0) = 0, \\ r\lambda_2^1(A_0) + \gamma_2^1(B_0) = 0, \\ r(r-1) + r\lambda_2^2(A_0) + \gamma_2^2(B_0) = 0, \\ r\lambda_1^2(A_0) + \gamma_1^2(B_0) = 0, \end{cases}$$

care se scrie în formă echivalentă

$$(4'') \quad \begin{cases} r(r-1) + r\lambda_1^1(A_0) + \gamma_1^1(B_0) = 0, \\ r(\lambda_1^1(A_0) - \lambda_2^2(A_0)) + \gamma_1^1(B_0) - \gamma_2^2(B_0) = 0, \\ r\lambda_2^1(A_0) + \gamma_2^1(B_0) = 0, \\ r\lambda_1^2(A_0) + \gamma_1^2(B_0) = 0. \end{cases}$$

Dacă:

$$r_0 = \frac{\gamma_2^2(B_0) - \gamma_1^1(B_0)}{\lambda_1^1(A_0) - \lambda_2^2(B_0)}, \qquad \gamma_2^1(B_0) = -r_0\lambda_2^1(A_0),$$

$$\gamma_1^2(B_0) = -r_0\lambda_1^2(A_0), \quad r_0(r_0 - 1) + r_0\lambda_1^1(A_0) + \gamma_1^1(B_0) = 0$$

atunci sistemul (4′) este compatibil şi deci ipoteza (I_1) îndeplinită, cu λ_1^1, λ_2^2, γ_1^1, γ_2^2 arbitrari.

Dacă în plus $|\lambda_{1i}^1|$ şi $|\lambda_{2i}^2|$ se aleg suficient de mici, atunci $|r_0|$ devine suficient de mare, încît operatorii T_n să fie contractanţi. În acest mod putem satisface şi ipoteza (I_2).

Am demonstrat astfel că dacă $E = C^2$ se pot construi ecuaţii de forma (1) care să satisfacă ipotezele (I_1) şi (I_2).

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Institutul politehnic, Iaşi,
Catedra de matematică

BIBLIOGRAFIE

1. Haimovici A., *Ecuaţii diferenţiale şi ecuaţii integrale.* E.D.P., Buc., 1965.
2. Gheorghiu N., *Introducere în analiza funcţională.* Ed. Acad. R.S.R., Buc., 1974.

ÉQUATIONS DIFFÉRENTIELLES À SINGULARITÉS DANS DES ESPACES DE BANACH
(Résumé)

On montre que l'équation

1) $$y'' + \varphi_1(p_1(z), y') + \varphi_2(p_2(z), y) = 0$$

où :

a) $$\varphi_1 : E \times E \to E \quad \text{et} \quad \varphi_2 : E \times E \to E,$$

— E un espace de Banach sur C (le champ complex) — sont billiniaires et

b) $$p_1 : U \setminus \{0\} \to E, \quad p_1(z) = \frac{1}{z} \sum_{i=0}^{\infty} A_i z^i, \quad A_i \in E,$$

$$p_2 : U \setminus \{0\} \to E, \quad p_2(z) = \frac{1}{z^2} \sum_{i=0}^{\infty} B_i z^i, \quad B_i \in E,$$

U un voisinage de $0 \in C$, a des solutions $y = z^r \sum_{i=0}^{\infty} C_i z^i$, $C_0 \neq 0$, $C_i \in E$, en supposant (I_1) et (I_2).

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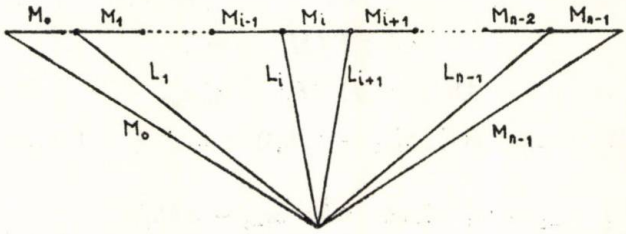
CIRCULAŢIE ÎN GRAFURI TRASATE ARBITRAR GENERALIZATE

DE

VIOREL GHIUŞ

D e f i n i ţ i e. Numim graf evantai, un graf lob de forma:



unde M_i $(i = 0, 1,..., n-1)$ şi L_j $(j = 1, 2,..., n-1)$ sînt muchii sau drumuri simple.

Vrem să studiem circulaţia în acest graf neorientat, ştiind că:

$$(1) \qquad \begin{cases} 0 \leqslant m_i \leqslant F(M_i) \leqslant c(M_i), & i = 0, 1,..., n-1 \\ 0 \leqslant l_j \leqslant F(L_j) \leqslant c(L_j), & j = 1, 2,..., n-1. \end{cases}$$

Definim cantităţile:

$$m'_{n-1} = m_{n-1}$$
$$m'_i = \max(m_i, l_{i+1} + m'_{i+1}), \quad i = 0, 1,..., n-2;$$
$$m''_{n-1} = c(M_{n-1})$$
$$m''_j = \min(c(M_j), m''_{j+1} + c(L_{j+1})), \quad j = 0,..., n-2.$$

Observaţia 1. $m_i \leqslant m'_i$, $\quad i = 0, 1,..., n-1$.

Observaţia 2. $m''_i \leqslant c(M_i)$, $\quad i = 0, 1,..., n-1$.

Observaţia 3. $m'_i \geqslant m'_{i+1}$, $\quad i = 0, 1,..., n-2$.

Pentru a dovedi ultima observaţie, vom deosebi două cazuri:

a) dacă $m_i > l_{i+1} + m'_{i+1}$ rezultă $m'_i = m_i$ şi $m_i > m'_{i+1}$, adică $m'_i > m'_{i+1}$.

b) dacă $m_i \leqslant l_{i+1} + m'_{i+1}$ rezultă $m'_i = l_{i+1} + m'_{i+1}$, adică $m'_i \geqslant m'_{i+1}$.

Observaţia 4. În general nu este îndeplinită condiţia $m'_i \leqslant m''_i$.

T e o r e m a 1. *Condiţia necesară şi suficientă ca într-un graf evantai să existe o circulaţie de forma*

$$F(M_i) = F(M_{i+1}) + F(L_{i+1}), \qquad i = 0, 1, \ldots, n-2$$

satisfăcînd condiţiile (1) *este ca*

$$m'_i \leqslant m''_i, \quad i = 0, 1, \ldots, n-1.$$

N e c e s i t a t e a. Evident $m'_{n-1} \leqslant F(M_{n-1}) \leqslant m''_{n-1}$. Presupunem că $m'_{i+1} \leqslant F(M_{i+1}) \leqslant m''_{i+1}$.

Din $F(M_i) = F(M_{i+1}) + F(L_{i+1})$ rezultă :

$$m'_{i+1} + l_{i+1} \leqslant F(M_i) \leqslant m''_{i+1} + c(L_{i+1});$$

în plus

$$m_i \leqslant F(M_i) \leqslant c(M_i).$$

Deci $\max(m_i, m'_{i+1} + l_{i+1}) \leqslant F(M_i) \leqslant \min(c(M_i), m''_{i+1} + c(L_{i+1}))$, sau

$$m'_i \leqslant F(M_i) \leqslant m''_i.$$

S u f i c i e n ţ a. Definim $F(M_0) = m'_0 \alpha_0$,

$$F(M_i) = \max(m'_i, \alpha_{i-1} - c(L_i)) = \alpha_i, \quad i = 1, 2, \ldots, n-2$$

şi

$$F(L_i) = \begin{cases} \alpha_{i-1} - m'_i & \text{dacă } m'_i \geqslant \alpha_{i-1} - c(L_i), \\ c(L_i) & \text{dacă } m'_i < \alpha_{i-1} - c(L_i), \end{cases} \qquad i = 1, 2, \ldots, n-1.$$

Se verifică, că $m'_1 \leqslant F(M_1) \leqslant m''_1$ şi $l_1 \leqslant F(L_1) \leqslant c(L_1)$.

Presupunem că $m'_{i-1} \leqslant F(M_{i-1}) = \alpha_{i-1} \leqslant m''_{i-1}$.

Vom deosebi cazurile : A) $m'_i \geqslant \alpha_{i-1} - c(L_i)$ şi B) $m'_i < \alpha_{i-1} - c(L_i)$.

Pe de o parte :

$$\left.\begin{array}{l} \text{A) } \alpha_i = m'_i \leqslant m''_i, \\ \text{B) } \alpha_i = \alpha_{i-1} - c(L_i) > m'_i; \\ \text{cum } \alpha_{i-1} \leqslant m''_{i-1} \text{ şi } m''_{i-1} \leqslant m''_i + c(L_i) \\ \text{avem } \alpha_{i-1} - c(L_i) \leqslant m''_{i-1} - c(L_i) \leqslant m''_i \end{array}\right\} \Rightarrow m'_i \leqslant \alpha_i \leqslant m''_i.$$

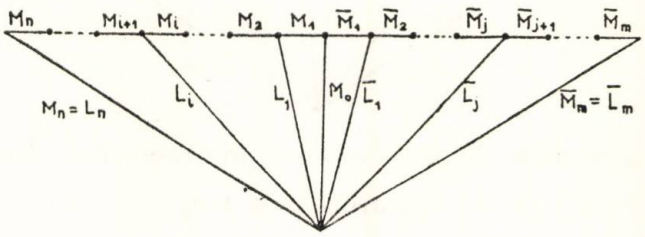
Pe de altă parte :

$$\left.\begin{array}{l} \text{A) } F(L_i) = \alpha_{i-1} - m'_i \leqslant c(L_i); \\ \text{cum } m'_{i-1} \geqslant l_i + m'_i \text{ sau } m'_{i-1} - m'_i \geqslant l_i \\ \text{avem } F(L_i) = \alpha_{i-1} - m'_i \geqslant m'_{i-1} - m'_i \geqslant l_i, \\ \text{B) } F(L_i) = c(L_i) \end{array}\right\} \Rightarrow l_i \leqslant F(L_i) \leqslant c(L_i).$$

Este satisfăcută relaţia $F(M_i) = F(M_{i+1}) + F(L_{i+1})$, $i = 0, 1, 2, \ldots, n-2$.

Se verifică în sfîrşit că $F(M_0) = \sum_{i=1}^{n} F(L_i)$.

Fie un graf evantai de forma :



Vrem să studiem circulaţia în acest graf neorientat, ştiind că :

$$(2)\begin{cases} 0 \leqslant m_i \leqslant F(M_i) \leqslant c(M_i), & i = 0, 1, \ldots, n, \\ 0 \leqslant l_i \leqslant F(L_i) \leqslant c(L_i), & i = 1, 2, \ldots, n-1, \\ 0 \leqslant \overline{m}_j \leqslant F(\overline{M}_j) \leqslant c(\overline{M}_j), & j = 1, 2, \ldots, m, \\ 0 \leqslant \overline{l}_j \leqslant F(\overline{L}_j) \leqslant c(\overline{L}_j), & j = 1, 2, \ldots, m, \end{cases} \quad \begin{matrix} (m_n = l_n \text{ şi } c(M_n) = c(L_n)), \\ \\ (\overline{m}_n = \overline{l}_n \text{ şi } c(\overline{M}_m) = c(\overline{L}_m)) \end{matrix}$$

Definim cantităţile :

$$\begin{array}{lcl} m'_n = m_n & \text{şi} & m''_n = c(M_n) \\ m'_i = \max(m_i, 1 + m'_{i+1}) & & m''_i = \min(c(M_i), m''_{i+1} + c(L_i)), \quad i = 1, 2, \ldots, n-1, \\ \overline{m}'_m = \overline{m}_m & & \overline{m}''_m = c(\overline{M}_m) \\ \overline{m}'_j = \max(\overline{m}_j, t_j + \overline{m}_{j+1}) & \text{şi} & \overline{m}''_j = \min(c(\overline{M}_j), \overline{m}''_{j+1} + c(\overline{L}_j)), j = 1, 2, \ldots, m-1; \\ m'_0 = \max(m_0, m'_1 + \overline{m}'_1) & & m''_0 = \min(c(M_0), m''_1 + \overline{m}''_1). \end{array}$$

T e o r e m a 2. *Condiţia necesară şi suficientă ca într-un graf evantai să existe o circulaţie de forma :*

$$\begin{array}{lll} F(M_0) = F(M_1) + F(M_1), & & \\ F(M_i) = F(M_{i+1}) + F(L_i) & \textit{pentru} & i = 1, 2, \ldots, n \\ F(\overline{M}_j) = F(\overline{M}_{j+1}) + F(\overline{L}_j) & \textit{pentru} & j = 1, 2, \ldots m \end{array}$$

satisfăcînd condiţiile (2), *este ca*

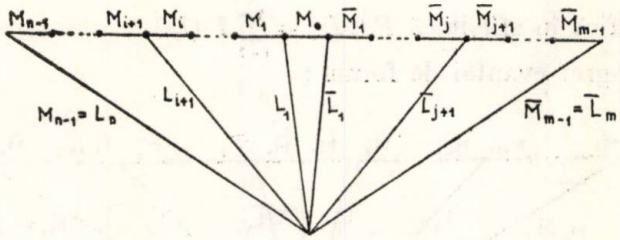
$$m'_i \leqslant m''_i \quad \textit{pentru} \quad i = 0, 1, 2, \ldots, n$$

şi

$$\overline{m}'_j \leqslant \overline{m}''_j \quad \textit{pentru} \quad j = 1, 2, \ldots, m.$$

D e m o n s t r a ţ i a este analogă cu cea de la teorema 1.

Fie un graf evantai de forma :



Vrem să studiem circulaţia în acest graf neorientat, ştiind că

$$(3)\begin{cases} 0 \leqslant m_i \leqslant F(M_i) \leqslant c(M_i) & \text{pentru } i = 0, 1, 2, \ldots, n-1 \\ 0 \leqslant \overline{m}_j \leqslant F(\overline{M}_j) \leqslant c(\overline{M}_j) & \text{pentru } j = 1, 2, \ldots, m-1 \\ 0 \leqslant l_i \leqslant F(L_i) \leqslant c(L_i) & \text{pentru } i=1, \ldots, n \ (l_n = m_{n-1} \text{ şi } c(L_n) = c(M_n)). \\ 0 \leqslant \overline{l}_j \leqslant F(\overline{L}_j) \leqslant c(\overline{L}_j) & \text{pentru } j=1, \ldots, m \ (\overline{l}_m = \overline{m}_{m-1} \text{ şi } c(\overline{L}_m) = c(\overline{M}_m)). \end{cases}$$

Definim cantităţile :

$$m'_{n-1} = m_{n-1} \qquad m''_{n-1} = c(M_{n-1})$$

$$m'_i = \max(m_i, l_{i+1} + m'_{i+1}) \quad \text{şi} \quad m''_i = \min(c(M_i), c(L_{i+1}) + m''_{i+1}),$$

$$\overline{m}'_{m-1} = \overline{m}_{m-1} \qquad \overline{m}''_{m-1} = c(\overline{M}_{m-1}), \quad i = 1, 2, \ldots, n-2$$

$$\overline{m}'_j = \max(\overline{m}_j, \overline{l}_{j+1} + \overline{m}'_{j+1}) \qquad \overline{m}''_j = \min(c(\overline{M}_j), c(\overline{L}_{j+1}) + \overline{m}''_{j+1}),$$

şi

$$m'_0 = \max(m_0, m'_1 + l_1, \overline{m}'_1 + l_1) \qquad m''_0 = \min(c(M_0), m''_1 + c(L_1), \overline{m}''_1 + c(\overline{L}_1))$$

$$j = 1, \ldots, m-2.$$

T e o r e m a 3. *Condiţia necesară şi suficientă ca într-un graf evantai să existe o circulaţie de forma :*

$$F(M_1) + F(L_1) = F(M_0) = F(\overline{M}_1) + F(\overline{L}_1)$$

$$F(M_i) = F(M_{i+1}) + F(L_{i+1}) \qquad i = 1, 2, \ldots, n-1$$

$$F(\overline{M}_j) = F(\overline{M}_{j+1}) + F(\overline{L}_{j+1}) \qquad j = 1, 2, \ldots, m-1$$

satisfăcînd condiţiile (3) *este ca*

$$m'_i \leqslant m''_i, \qquad i = 0, 1, \ldots, n-1,$$

şi

$$\overline{m}'_j \leqslant \overline{m}''_j, \qquad j = 1, 2, \ldots, m-1.$$

D e m o n s t r a ţ i a este analogă cu cea de la teorema 1.

Presupunem că avem un graf stea G_a şi un nod b diferit de orice nod al lui G_a. Dacă unim b cu toate nodurile lui G_a prin muchii (drumuri simple), se formează un graf scul.

Se ştie [3] că orice graf trasat arbitrar se obţine dintr-un arbore, ale cărui noduri se unesc prin muchii (drumuri simple) cu un nou nod.

Deci, orice subgraf, format din noul nod şi un nod neterminal oarecare al arborelui, este un subgraf scul. Nodurile neterminale ale arborelui şi noul nod le vom numi noduri esenţiale.

În general ne putem pune problema existenţei unei circulaţii într-un graf evantai în care oricare două noduri esenţiale, adiacente, determină un graf trasat arbitrar generalizat.

T e o r e m a 4. *Dacă într-un graf trasat arbitrar generalizat G, există o circulaţie, atunci această circulaţie se poate descompune într-o sumă de circulaţii : o circulaţie în graful evantai la care se poate reduce G şi la cîte o circulaţie în fiecare graf scul.*

Reciproca este de asemenea adevărată.

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Institutul politehnic, Iaşi,
Catedra de matematică

B I B L I O G R A F I E

1. F o r d L. R., F u l k e r s o n D. R., *Flows in Networks.* Princeton University Press, 1962.
2. G h i u ş V., *Circulaţie în grafuri trasate arbitrar.* Bul. Inst. polit. Iaşi, XXII (XXVI), *3—4*, s. I (1976).
3. O r e O., *Theory of Graphs.* American Society Colloquium Publications, Providence, Rhode Island, vol XXXVIII, 1962.

LA CIRCULATION DANS LES GRAPHES TRACÉS ARBITRAIREMENT, GÉNÉRALISÉS

(Résumé)

On défini la notion de graphe éventail et on étudie le problème de la circulation dans tel graphe. En utilisant ce fait on peut poser et résoudre le problème de la circulation dans des graphes tracés arbitrairement généralisés à l'aide de la circulation dans des graphes tracés arbitrairement de deux sommets et des graphes éventails.

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ASUPRA VARIETĂŢILOR COMPLEXE CU CONEXIUNE NESIMETRICĂ DEPENDENTĂ DE DIRECŢIE

DE

I. GHINEA

1. În această lucrare, care continuă şi extinde rezultatele obţinute în [9], se stabilesc unele relaţii pentru varietăţi complexe cu conexiune nesimetrică, depinzînd de direcţie, folosind în acest scop o conexiune simetrică asociată şi tensorii simetrici de curbură asociaţi.

Fie X_{2n} o varietate reală $2n$-dimensională, raportată la coordonatele reale (x^j, y^j), $j = 1, 2, \ldots, n$. Fiecărui punct P din X_{2n}, de coordonate (x^j, y^j) îi corespund numerele complexe

$$z^j = x^j + i\, y^j, \qquad z^{j*} = x^j - i\, y^j, \qquad (i^2 = -1) \tag{1}$$

unde z^j se numesc coordonatele complexe ale lui P.

După cum se ştie [1], [2], [3], varietatea X_{2n} raportată la sistemul de coordonate complexe se numeşte spaţiu complex sau varietate complexă şi se notează cu C_n, iar o transformare de coordonate pe C_n este de forma

$$\bar{z}^j = \bar{z}\,(z^j), \qquad \bar{z}^{j*} = \bar{z}^{j*}\,(z^{j*}). \tag{2}$$

Să considerăm acum, după H. Rund [4], [5], [6] că pe varietatea complexă C_n este definită o conexiune Γ depinzînd de direcţie, ai cărei coeficienţi

$$\Gamma_{h\,k}^{\;j}\,(z^l, z^{l*}, \dot{z}^l, \dot{z}^{l*}), \qquad \Gamma_{h^*\,k^*}^{\;j^*}\,(z^l, z^{l*}, \dot{z}^l, \dot{z}^{l*}) \tag{3}$$

satisfac următoarea lege de transformare prin (1):

$$B_{m\,k}^{\;l} = B_t^l\, \bar{\Gamma}_{m\,k}^{\;t} - \Gamma_{p\,h}^{\;l}\, B_m^p\, B_k^h, \tag{4}$$

$$B_{m^*\,k^*}^{l^*} = B_{t^*}^{l^*}\, \bar{\Gamma}_{m^*\,k^*}^{t^*} - \Gamma_{p^*\,k^*}^{l^*}\, B_{m^*}^{p^*}\, B_{k^*}^{h^*}, \tag{4'}$$

sau echivalenta sa

$$A_{l\,k}^{\;m} = A_t^m\, \Gamma_{l\,k}^{\;t} - \bar{\Gamma}_{r\,q}^{\;m}\, A_l^r\, A_k^q \tag{5}$$

şi conjugata respectivă, unde

$$\frac{\partial \bar{z}^j}{\partial z^h} = A^j_h(z), \qquad \frac{\partial \bar{z}^{j^*}}{\partial z^{h^*}} = A^{j^*}_{h^*}(z^*), \qquad \frac{\partial^2 \bar{z}^j}{\partial z^h \partial z^k} = A_h{}^j{}_k(z),$$

$$\frac{\partial z^h}{\partial \bar{z}^j} = B^h_j(\bar{z}), \qquad \frac{\partial z^{h^*}}{\partial \bar{z}^{j^*}} = B^{h^*}_{j^*}(\bar{z}^*), \qquad \frac{\partial^2 z^h}{\partial \bar{z}^j \partial \bar{z}^m} = B_j{}^h{}_m(\bar{z}).$$

Vom presupune în continuare că conexiunea Γ este nesimetrică.

Din condiţiile de integrabilitate a legii de transformare (4) şi (4') rezultă doi tensori de curbură [5], [6]; primul are componentele

$$K_m{}^j{}_{hk^*} = \frac{\partial \Gamma_m{}^j{}_h}{\partial z^{k^*}} - \frac{\partial \Gamma_m{}^j{}_h}{\partial z^{l^*}} \Gamma^{l^*}_{k^*} \tag{6}$$

şi corespunzătorul său „stelat" (conjugat), în timp ce al doilea are componentele

$$K_h{}^j{}_{km} = \frac{\partial \Gamma_h{}^j{}_k}{\partial z^m} - \frac{\partial \Gamma_h{}^j{}_k}{\partial z^l} \Gamma^l_m - \left(\frac{\partial \Gamma_h{}^j{}_m}{\partial z^k} - \frac{\partial \Gamma_h{}^j{}_m}{\partial z^l} \Gamma^l_k \right) + \Gamma_p{}^j{}_m \Gamma_h{}^p{}_k - \Gamma_p{}^j{}_k \Gamma_h{}^p{}_m \tag{7}$$

şi corespunzătorul său „stelat" (conjugat).

2. Să stabilim acum unele relaţii pentru o varietate complexă C_n, cu conexiune generalizată Γ, nesimetrică, cu ajutorul conexiunii simetrice asociate S.

Prin analogie cu cazul spaţiilor reale, cu conexiune afină nesimetrică [8], [9] vom asocia conexiunii generalizate nesimetrice Γ, o conexiune simetrică S ai cărui coeficienţi sînt daţi de relaţiile

$$S_h{}^j{}_k = \frac{1}{2}(\Gamma_h{}^j{}_k + \Gamma_k{}^j{}_h) = \Gamma_{(h}{}^j{}_{k)}, \qquad S_{h^*}{}^{j^*}{}_{k^*} = \frac{1}{2}(\Gamma_{h^*}{}^{j^*}{}_{k^*} + \Gamma_{k^*}{}^{j^*}{}_{h^*}) = \Gamma_{(h^*}{}^{j^*}{}_{k^*)} \tag{8}$$

şi un tensor de torsiune Ω cu componentele

$$\Omega_h{}^j{}_k = \frac{1}{2}(\Gamma_h{}^j{}_k - \Gamma_k{}^j{}_h) = \Gamma_{[h}{}^i{}_{k]}, \qquad \Omega_{h^*}{}^{j^*}{}_{k^*} = \frac{1}{2}(\Gamma_{h^*}{}^{j^*}{}_{k^*} - \Gamma_{k^*}{}^{j^*}{}_{h^*}) = \Gamma_{[h^*}{}^{j^*}{}_{k^*]}. \tag{9}$$

Din (8) şi (9) obţinem

$$\Gamma_h{}^j{}_k = S_h{}^j{}_k + \Omega_h{}^j{}_k, \qquad \Gamma_{h^*}{}^{j^*}{}_{k^*} = S_{h^*}{}^{j^*}{}_{k^*} + \Omega_{h^*}{}^{j^*}{}_{k^*}. \tag{10}$$

Introducînd notaţiile [5], [6]

$$\Gamma^j_k = \Gamma_h{}^j{}_k \dot{z}^h, \qquad \Gamma^{j^*}_{k^*} = \Gamma_{h^*}{}^{j^*}{}_{k^*} \dot{z}^{h^*} \tag{11}$$

şi notaţii analoge pentru S şi Ω, rezultă următoarele relaţii

$$S^j_k = \frac{1}{2}(\Gamma^j_k + \Gamma_k{}^j{}_h \dot{z}^h), \tag{12} \qquad S^j_k \dot{z}^k = \Gamma^j_k \dot{z}^k, \tag{13}$$

respectiv

$$\Omega^j_k = \frac{1}{2}(\Gamma^j_k - \Gamma_k{}^j{}_h \dot{z}^h), \tag{14} \qquad \Omega^j_k \dot{z}^k = 0 \tag{15}$$

precum şi formele analoge pentru componentele „stelate" (conjugate). Din (12) şi (14) se obţine

$$S^j_k + \Omega^j_k = \Gamma^j_k\,, \qquad (16)$$

$$S^j_k - \Omega^j_k = \Gamma^j_{kh}\,\dot{z}^h\,. \qquad (17)$$

Fie

$$\Omega_{m}{}^{j}{}_{h/k^*} = \frac{\partial \Omega_{m}{}^{j}{}_{h}}{\partial z^{k^*}} - \frac{\partial \Omega_{m}{}^{j}{}_{h}}{\partial \dot{z}^{l^*}}\,\Gamma^{l^*}_{k^*}\,, \qquad (18)$$

derivata covariantă a tensorului de torsiune $\Omega_{m}{}^{j}{}_{h}$ în raport cu conexiunea nesimetrică Γ şi

$$\Omega_{m}{}^{j}{}_{k^*} = \frac{\partial \Omega_{m}{}^{j}{}_{h}}{\partial z^{k^*}} - \frac{\partial \Omega_{m}{}^{j}{}_{h}}{\partial \dot{z}^{l^*}}\,S^{l^*}_{k^*}\,, \qquad (19)$$

derivata covariantă a lui $\Omega_{m}{}^{j}{}_{h}$ în raport cu conexiunea simetrică S.

Dacă în (18) se ţine seama de (19) se obţine relaţia

$$\Omega_{m}{}^{j}{}_{h/k^*} = \Omega_{m}{}^{j}{}_{h,k^*} - \frac{\partial \Omega_{m}{}^{j}{}_{h}}{\partial \dot{z}^{l^*}}\,\Omega^{l^*}_{k^*}\,, \qquad (20)$$

care exprimă legătura între derivatele covariante ale tensorului $\Omega_{m}{}^{j}{}_{h}$ în raport cu cele două conexiuni Γ şi S.

Dacă se introduce (10) în (6) rezultă

$$K_{m}{}^{j}{}_{hk^*} = S_{m}{}^{j}{}_{hk^*} + T_{m}{}^{j}{}_{hk^*} \qquad (21)$$

şi analog pentru corespunzătorul „stelat" (conjugat), unde

$$S_{m}{}^{j}{}_{hk^*} = \frac{\partial S_{m}{}^{j}{}_{h}}{\partial z^{k^*}} - \frac{\partial S_{m}{}^{j}{}_{h}}{\partial \dot{z}^{l^*}}\,S^{l^*}_{k^*} \qquad (22)$$

şi conjugata sa, sînt componentele primului tensor de curbură corespunzător conexiunii simetrice S, care este un tensor simetric în raport cu indicii m şi h, iar

$$T_{m}{}^{j}{}_{hk^*} = \left(\frac{\partial \Omega_{m}{}^{j}{}_{h}}{\partial z^{k^*}} - \frac{\partial \Omega_{m}{}^{j}{}_{h}}{\partial \dot{z}^{l^*}}\,S^{l^*}_{k^*}\right) - \left(\frac{\partial S_{m}{}^{j}{}_{h}}{\partial \dot{z}^{l^*}} + \frac{\partial \Omega_{m}{}^{j}{}_{h}}{\partial \dot{z}^{l^*}}\right)\Omega^{l^*}_{k^*} \qquad (23)$$

şi conjugata sa, sînt componentele unui tensor, cu valenţele indicate de poziţia indicilor.

Din (23) şi (19) rezultă următoarea expresie pentru $T_{m}{}^{j}{}_{hk^*}$:

$$T_{m}{}^{j}{}_{hk^*} = \Omega_{m}{}^{j}{}_{h,k^*} - \left(\frac{\partial S_{m}{}^{j}{}_{h}}{\partial \dot{z}^{l^*}} + \frac{\partial \Omega_{m}{}^{j}{}_{h}}{\partial \dot{z}^{l^*}}\right)\Omega^{l^*}_{k^*}\,. \qquad (24)$$

Dacă în (24) se schimbă m cu h, prin adunare şi scădere se obţine

$$T_{m}{}^{j}{}_{hk^*} + T_{h}{}^{j}{}_{mk^*} = -\,2\,\frac{\partial S_{h}{}^{j}{}_{m}}{\partial \dot{z}^{l^*}}\,\Omega^{l^*}_{k^*}\,, \qquad (25)$$

$$T_{m\,hk^*}^{\;\;j} - T_{h\,mk^*}^{\;\;j} = 2\Omega_{m\,h,k^*}^{\;\;j} - 2\frac{\partial \Omega_{m\,h}^{\;\;j}}{\partial \dot{z}^{l^*}}\Omega_{k^*}^{l^*} \tag{26}$$

iar din (26) şi (20)

$$T_{m\,hk^*}^{\;\;j} - T_{h\,mk^*}^{\;\;j} = 2\Omega_{m\,h/k^*}^{\;\;j}\,. \tag{27}$$

Dacă se multiplică (24) cu $\dot{z}^{k^*}$ şi se ţine seama de (15) se obţine

$$T_{m\,hk^*}^{\;\;j}\,\dot{z}^{k^*} = \Omega_{m\,h,k^*}^{\;\;j}\,\dot{z}^{k^*}. \tag{28}$$

Prin înmulţirea lui (25) cu $\dot{z}^{k^*}$ rezultă

$$(T_{m\,hk^*}^{\;\;j} + T_{h\,mk^*}^{\;\;j})\,\dot{z}^{k^*} = 0. \tag{29}$$

Din (21), ţinînd seama de (25) respectiv (26) sau (27) rezultă

$$K_{m\,hk^*}^{\;\;j} + K_{h\,mk^*}^{\;\;j} = 2S_{m\,hk^*}^{\;\;j} - 2\frac{\partial S_{h\,m}^{\;\;j}}{\partial \dot{z}^{l^*}}\Omega_{k^*}^{l^*} \tag{30}$$

şi

$$K_{m\,hk^*}^{\;\;j} - K_{h\,mk^*}^{\;\;j} = 2\Omega_{m\,h,k^*}^{\;\;j} - 2\frac{\partial \Omega_{m\,h}^{\;\;j}}{\partial \dot{z}^{l^*}}\Omega_{k^*}^{l^*} = 2\Omega_{m\,h/k^*}^{\;\;j}\,. \tag{31}$$

Să considerăm acum derivatele covariante ale lui $K_{r\,hk^*}^{\;\;m}$ în raport cu Γ ([5], [6])

$$K_{r\,hk^*/j}^{\;\;m} = \frac{\partial K_{r\,hk^*}^{\;\;m}}{\partial z^j} - \frac{\partial K_{r\,hk^*}^{\;\;m}}{\partial \dot{z}^l}\Gamma_j^l + K_{r\,hk^*}^{\;\;p}\Gamma_{p\,j}^{\;\;m} - K_{p\,hk^*}^{\;\;m}\Gamma_{r\,j}^{\;\;p} - K_{r\,pk^*}^{\;\;m}\Gamma_{h\,j}^{\;\;p} \tag{32}$$

şi în raport cu S

$$K_{r\,hk^*,j}^{\;\;m} = \frac{\partial K_{r\,hk^*}^{\;\;m}}{\partial z^j} - \frac{\partial K_{r\,hk^*}^{\;\;m}}{\partial \dot{z}^l}S_j^l + K_{r\,hk^*}^{\;\;p}S_{p\,j}^{\;\;m} - K_{p\,hk^*}^{\;\;m}S_{r\,j}^{\;\;p} - K_{r\,pk^*}^{\;\;m}S_{h\,j}^{\;\;p}. \tag{33}$$

Dacă se ţine seama de (10) şi (33) în (32) rezultă

$$K_{r\,hk^*/j}^{\;\;m} = K_{r\,hk^*,j}^{\;\;m} - \frac{\partial K_{r\,hk^*}^{\;\;m}}{\partial \dot{z}^l}\Omega_j^l + K_{r\,hk^*}^{\;\;p}\Omega_{p\,j}^{\;\;m} - K_{p\,hk^*}^{\;\;m}\Omega_{r\,j}^{\;\;p} - K_{r\,pk^*}^{\;\;m}\Omega_{h\,j}^{\;\;p} \tag{34}$$

care stabileşte o legătură între derivatele covariante ale tensorului de curbură $K_{r\,hk^*}^{\;\;m}$ în raport cu cele două conexiuni Γ şi S.

Prin permutarea indicilor j, k^*, h în relaţia (34) şi sumare, se obţine

$$\begin{aligned} K_{r\,jk^*/h}^{\;\;m} + K_{r\,hk^*/j}^{\;\;m} + K_{r\,hj/k^*}^{\;\;m} &= K_{r\,jk^*,h}^{\;\;m} + K_{r\,hk^*,j}^{\;\;m} + K_{r\,hj,k^*}^{\;\;m} - \\ &- \left(\frac{\partial K_{r\,jk^*}^{\;\;m}}{\partial \dot{z}^l}\Omega_h^l + \frac{\partial K_{r\,hk^*}^{\;\;m}}{\partial \dot{z}^l}\Omega_j^l + \frac{\partial K_{r\,hj}^{\;\;m}}{\partial \dot{z}^{l^*}}\Omega_{k^*}^{l^*}\right) + \\ &+ (K_{r\,jk^*}^{\;\;p}\Omega_{p\,h}^{\;\;m} - K_{p\,jk^*}^{\;\;m}\Omega_{r\,h}^{\;\;p}) + (K_{r\,hk^*}^{\;\;p}\Omega_{p\,j}^{\;\;m} - K_{p\,hk^*}^{\;\;m}\Omega_{r\,j}^{\;\;p}). \end{aligned} \tag{35}$$

Relaţii analoge se pot obţine din (32) şi (33) prin permutarea indicilor j, k^*, h şi sumare.

Fie ([5], [6])

$$\Omega_h{}^j{}_{k/m} = \frac{\partial \Omega_h{}^j{}_k}{\partial z^m} - \frac{\partial \Omega_h{}^j{}_k}{\partial \dot{z}^l} \Gamma^l_m + \Omega_h{}^p{}_k \Gamma_p{}^j{}_m - \Omega_p{}^j{}_k \Gamma_h{}^p{}_m - \Omega_h{}^j{}_p \Gamma_k{}^p{}_m \tag{36}$$

derivata covariantă a tensorului de torsiune $\Omega_h{}^j{}_k$ în raport cu conexiunea nesimetrică Γ şi

$$\Omega_h{}^j{}_{k,m} = \frac{\partial \Omega_h{}^j{}_k}{\partial z^m} - \frac{\partial \Omega_h{}^j{}_k}{\partial \dot{z}^l} S^l_m + \Omega_h{}^p{}_k S_p{}^j{}_m - \Omega_p{}^j{}_k S_h{}^j{}_m - \Omega_h{}^j{}_p S_k{}^p{}_m \tag{37}$$

derivata covariantă a lui $\Omega_h{}^j{}_k$ în raport cu conexiunea simetrică S.

Dacă în (36) ţinem seama de (10) şi (37) obţinem relaţia

$$\Omega_h{}^j{}_{k/m} = \Omega_h{}^j{}_{k,m} - \frac{\partial \Omega_h{}^j{}_k}{\partial \dot{z}^l} \Omega^l_m + \Omega_h{}^p{}_k \Omega_p{}^j{}_m - \Omega_p{}^j{}_k \Omega_h{}^p{}_m - \Omega_h{}^j{}_p \Omega_k{}^p{}_m \tag{38}$$

care exprimă legătura între derivatele covariante ale tensorului $\Omega_m{}^j{}_h$ în raport cu cele două conexiuni Γ şi S.

Din (7) şi (10) rezultă

$$K_h{}^j{}_{km} = S_h{}^j{}_{km} + T_h{}^j{}_{km} \tag{39}$$

unde

$$S_h{}^j{}_{km} = \frac{\partial S_h{}^j{}_k}{\partial z^m} - \frac{\partial S_h{}^j{}_k}{\partial \dot{z}^l} S^l_m - \left(\frac{\partial S_h{}^j{}_m}{\partial z^k} - \frac{\partial S_h{}^j{}_m}{\partial \dot{z}^l} S^l_k \right) + S_p{}^j{}_m S_h{}^p{}_k - S_p{}^j{}_k S_h{}^p{}_m \tag{40}$$

este al doilea tensor de curbură corespunzător conexiunii simetrice S, tensor care este antisimetric în indicii k şi m, iar

$$\begin{aligned} T_h{}^j{}_{km} = & \left(\frac{\partial \Omega_h{}^j{}_k}{\partial z^m} - \frac{\partial \Omega_h{}^j{}_k}{\partial \dot{z}^l} S^l_m \right) - \left(\frac{\partial \Omega_h{}^j{}_m}{\partial z^k} - \frac{\partial \Omega_h{}^j{}_m}{\partial \dot{z}^l} S^l_k \right) + \left(\frac{\partial S_h{}^j{}_m}{\partial \dot{z}^l} \Omega^l_k - \right. \\ & \left. - \frac{\partial S_h{}^j{}_k}{\partial \dot{z}^l} \Omega^l_m \right) + \left(\frac{\partial \Omega_h{}^j{}_m}{\partial \dot{z}^l} \Omega^l_k - \frac{\partial \Omega_h{}^j{}_k}{\partial \dot{z}^l} \Omega^l_m \right) + (\Omega_p{}^j{}_m S_h{}^p{}_k - \Omega_p{}^j{}_k S_h{}^p{}_m + \\ & + S_p{}^j{}_m \Omega_h{}^p{}_k - S_p{}^j{}_k \Omega_h{}^p{}_m) + (\Omega_p{}^j{}_m \Omega_h{}^p{}_k - \Omega_p{}^j{}_k \Omega_h{}^p{}_m) \end{aligned} \tag{41}$$

sînt componentele unui tensor, cu valenţele indicate de poziţia indicilor, tensor care este antisimetric în raport cu indicii k şi m.

Dacă punem în evidenţă în expresia lui $T_h{}^j{}_{km}$ dată de (41) derivatele covariante $\Omega_h{}^j{}_{k,m}$ şi $\Omega_h{}^j{}_{m,k}$ date de (37), rezultă relaţia

$$\begin{aligned} T_h{}^j{}_{km} = & \Omega_h{}^j{}_{k,m} - \Omega_h{}^j{}_{m,k} + \left(\frac{\partial S_h{}^j{}_m}{\partial \dot{z}^l} + \frac{\partial \Omega_h{}^j{}_m}{\partial \dot{z}^l} \right) \Omega^l_k - \left(\frac{\partial S_h{}^j{}_k}{\partial \dot{z}^l} + \frac{\partial \Omega_h{}^j{}_k}{\partial \dot{z}^l} \right) \Omega^l_m + \\ & + \Omega_p{}^j{}_m \Omega_h{}^p{}_k - \Omega_p{}^j{}_k \Omega_h{}^p{}_m , \end{aligned} \tag{42}$$

care exprimă tensorul $T_h{}^i{}_{km}$ cu ajutorul tensorilor de curbură şi al conexiunii simetrice.

Dacă în (42) ţinem seama de (16) obţinem

$$(43)\quad T_h{}^j{}_{km} = \Omega_h{}^j{}_{k,m} - \Omega_h{}^j{}_{m,k} + \frac{\partial \Gamma_h{}^j{}_m}{\partial z^l}\Omega_k^l - \frac{\partial \Gamma_h{}^j{}_k}{\partial z^l}\Omega_m^l + \Omega_p{}^j{}_m \Omega_h{}^p{}_k - \Omega_p{}^j{}_k \Omega_h{}^p{}_m .$$

Prin permutarea circulară a indicilor inferiori ai lui $S_h{}^j{}_{km}$ din relaţia (40) rezultă

$$(44)\qquad S_h{}^j{}_{km} + S_k{}^j{}_{mh} + S_m{}^j{}_{hk} = 0.$$

Dacă permutăm circular indicii inferiori în relaţia (39) şi sumăm, ţinînd seama şi de (44) rezultă

$$(45)\qquad K_h{}^j{}_{km} + K_k{}^j{}_{mh} + K_m{}^j{}_{hk} = T_h{}^j{}_{km} + T_k{}^j{}_{mh} + T_m{}^j{}_{hk},$$

care constituie o relaţie între tensorul de curbură şi tensorul $T_h{}^j{}_{km}$ independentă de tensorul $S_h{}^j{}_{km}$.

Dacă permutăm acum circular indicii h, k, m în (42) şi sumăm obţinem relaţia

$$(46)\quad \begin{aligned} T_h{}^j{}_{km} + T_k{}^j{}_{mh} + T_m{}^j{}_{hk} &= 2\,(\Omega_h{}^j{}_{k,m} + \Omega_k{}^j{}_{m,h} + \Omega_m{}^j{}_{h,k}) - \\ &- 2\left(\frac{\partial \Omega_h{}^j{}_k}{\partial z^l}\Omega_m^l + \frac{\partial \Omega_k{}^j{}_m}{\partial z^l}\Omega_h^l + \frac{\partial \Omega_m{}^j{}_h}{\partial z^l}\Omega_k^l\right) + 2\,(\Omega_p{}^j{}_h\,\Omega_k{}^p{}_m + \\ &+ \Omega_p{}^j{}_k\,\Omega_m{}^p{}_h + \Omega_p{}^j{}_m\,\Omega_h{}^p{}_k), \end{aligned}$$

care exprimă o legătură între tensorul $T_h{}^j{}_{km}$ şi $\Omega_h{}^j{}_k$.

Din (45) şi (46) rezultă

$$(47)\quad \begin{aligned} K_h{}^j{}_{km} + K_k{}^j{}_{mh} + K_m{}^j{}_{hk} &= 2\,(\Omega_h{}^j{}_{k,m} + \Omega_k{}^j{}_{m,h} + \Omega_m{}^j{}_{h,k}) - \\ &- 2\left(\frac{\partial \Omega_h{}^j{}_k}{\partial z^l}\Omega_m^l + \frac{\partial \Omega_k{}^j{}_m}{\partial z^l}\Omega_h^l + \frac{\partial \Omega_m{}^j{}_h}{\partial z^l}\Omega_k^l\right) + 2\,(\Omega_p{}^j{}_h\,\Omega_k{}^p{}_m + \\ &+ \Omega_p{}^j{}_k\,\Omega_m{}^p{}_h + \Omega_p{}^j{}_m\,\Omega_h{}^p{}_k). \end{aligned}$$

Permutînd indicii de covarianţă h, k, m în (7) şi sumînd, dacă se ţine seama de (9) şi de expresia derivatei covariante $\Omega_h{}^j{}_{k/m}$ dată de (36) se găseşte relaţia

$$(48)\quad \begin{aligned} K_h{}^j{}_{km} + K_k{}^j{}_{mh} + K_m{}^j{}_{hk} &= 2\,(\Omega_h{}^j{}_{k/m} + \Omega_k{}^j{}_{m/h} + \Omega_m{}^j{}_{h/k}) - \\ &- 4\,(\Omega_p{}^j{}_h\,\Omega_m{}^p{}_k + \Omega_p{}^j{}_k\,\Omega_m{}^p{}_h + \Omega_p{}^j{}_m\,\Omega_h{}^p{}_k). \end{aligned}$$

Din (47) şi (48) rezultă

$$(49)\quad \begin{aligned} \Omega_h{}^j{}_{k/m} + \Omega_k{}^j{}_{m/h} + \Omega_m{}^j{}_{h/k} &= \Omega_h{}^j{}_{k,m} + \Omega_k{}^j{}_{m,h} + \Omega_m{}^j{}_{h,k} - \\ &- \left(\frac{\partial \Omega_h{}^j{}_m}{\partial z^l}\Omega_k^l + \frac{\partial \Omega_k{}^j{}_h}{\partial z^l}\Omega_m^l + \frac{\partial \Omega_m{}^j{}_k}{\partial z^l}\Omega_k^l\right) + 3\,(\Omega_p{}^j{}_k\,\Omega_h{}^p{}_m + \\ &+ \Omega_p{}^j{}_h\,\Omega_m{}^p{}_k + \Omega_p{}^j{}_m\,\Omega_k{}^p{}_h). \end{aligned}$$

(Acelaşi rezultat se obţine permutînd circular indicii inferiori în (38) şi sumînd.)

Fie

$$K_r{}^m{}_{jk/h} = \frac{\partial K_r{}^m{}_{jk}}{\partial z^h} - \frac{\partial K_r{}^m{}_{jk}}{\partial \dot{z}^l}\Gamma^l_h + K_r{}^p{}_{jk}\,\Gamma_p{}^m{}_h - K_p{}^m{}_{jk}\,\Gamma_r{}^p{}_h - \tag{50}$$
$$- K_r{}^m{}_{pk}\,\Gamma_j{}^p{}_h - K_r{}^m{}_{jp}\,\Gamma_k{}^p{}_h$$

derivata covariantă a lui $K_r{}^m{}_{jk}$ în raport cu conexiunea nesimetrică Γ şi

$$K_r{}^m{}_{jk,h} = \frac{\partial K_r{}^m{}_{jk}}{\partial z^h} - \frac{\partial K_r{}^m{}_{jk}}{\partial \dot{z}^l}S^l_h + K_r{}^p{}_{jk}\,S_p{}^m{}_h - K_p{}^m{}_{jk}\,S_r{}^p{}_h - \tag{51}$$
$$- K_r{}^m{}_{pk}\,S_j{}^p{}_h - K_r{}^m{}_{jp}\,S_k{}^p{}_h$$

derivata covariantă în raport cu conexiunea simetrică S.

Dacă se ţine seama de (10) şi (51) în (50) rezultă

$$K_r{}^m{}_{jk/h} = K_r{}^m{}_{jk,h} - \frac{\partial K_r{}^m{}_{jk}}{\partial \dot{z}^l}\Omega^l_h + K_r{}^p{}_{jk}\,\Omega_p{}^m{}_h - K_p{}^m{}_{jk}\,\Omega_r{}^m{}_h - \tag{52}$$
$$- K_r{}^m{}_{pk}\,\Omega_j{}^p{}_h - K_r{}^m{}_{jp}\,\Omega_k{}^p{}_h.$$

Prin permutarea indicilor j, k, h în relaţia (52) şi sumare se obţine

$$K_r{}^m{}_{jk/h} + K_r{}^m{}_{kh/j} + K_r{}^m{}_{hj/k} = K_r{}^m{}_{k,h} + K_r{}^m{}_{kh,j} + K_r{}^m{}_{hj,k} -$$
$$-\left(\frac{\partial K_r{}^m{}_{jk}}{\partial \dot{z}^l}\Omega^l_p + \frac{\partial K_r{}^m{}_{kh}}{\partial \dot{z}^l}\Omega^l_j + \frac{\partial K_r{}^m{}_{hj}}{\partial \dot{z}^l}\Omega^l_k\right) + (K_r{}^p{}_{jk}\,\Omega_p{}^m{}_h + K_r{}^p{}_{kh}\,\Omega_p{}^m{}_j + \tag{53}$$
$$+ K_r{}^p{}_{hj}\,\Omega_p{}^m{}_k) - (K_p{}^m{}_{jk}\,\Omega_r{}^p{}_h + K_p{}^m{}_{kh}\,\Omega_r{}^p{}_j + K_p{}^m{}_{hj}\,\Omega_r{}^p{}_k) +$$
$$+ 2\,(K_r{}^m{}_{pj}\,\Omega_k{}^p{}_h + K_r{}^m{}_{pk}\,\Omega_h{}^p{}_j + K_r{}^m{}_{ph}\,\Omega_j{}^p{}_k).$$

Relaţii analoge se pot obţine din (51) respectiv (52) prin permutarea circulară a indicilor j, k, h şi sumare.

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Institutul politehnic, Iaşi,
Catedra de matematică

BIBLIOGRAFIE

1. Yano K., *Differential Geometry of Complex and Almost Complex Spaces.* Pergamon Press, Oxford, 1965.
2. Gheorghiu Gh. Th., *Sur les espaces à structure complexe.* Rev. roum. de mathém. pures et appl., XV, *9* (1970).
3. Gheorghiu Gh. Th., *Sur les espaces de Kähler.* Boll. U.M.I. (4), **4**, 153–163 (1971).
4. Rund H., *Generalized Metrics of Complex Manifolds.* Math. Nachrichten, **34**, 55–77 (1967).
5. Rund H., *The Theory of Generalized Connections and Curvature of Complex Manifolds.* Joint Math. Coll., The University of S.A. and the University of the Witwatersrand, pp. 47–49, 1967.
6. Rund H., *The Curvature Theory of Direction Dependent Connections on Complex Manifolds.* Tensor N.S., **24** (1972).
7. Rund H., *The Differential Geometry of Finsler Spaces.* Springer Verlag, Berlin–Göttingen–Heidelberg, 1959.

8. Vrănceanu Gh., *Leçons de géométrie différentielle*. Vol. I, Éd. de l'Acad., Bucarest, 1957.
9. Ghinea I., *Asupra spaţiilor cu conexiune afină nesimetrică*. Bul. Inst. polit. Iaşi, XXIII (XXVII), *1–2*, s. I, 53—58 (1977).

ON COMPLEX MANIFOLDS WITH NONSYMMETRIC DIRECTION DEPENDENT CONNECTIONS

(Summary)

One establishes some relations for complex manifolds, with nonsymmetric direction dependent connections, using an associated symmetric connection and the associated symmetric curvature tensors.

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PROIECŢIA CENTRALĂ (PERSPECTIVA) PRIN CORESPONDENŢĂ DE OMOLOGIE AFINĂ

DE

I. RUSU

1. Introducere. Se consideră planele $[H]$ şi $[P]$ concurente după dreapta $(\Delta) \equiv (\delta)$. Planul $[H]$ espe planul reprezentării pe care obiectul considerat este figurat prin proiecţia sa ortogonală. Al doilea plan $[P]$ se numeşte planul de proiecţie, tabloul sau ecranul pe care se construieşte proiecţia centrală sau perspectiva obiectului. Dreapta de intersecţie (Δ) dintre aceste plane se numeşte în acest caz, axă de omologie, fig. 1.

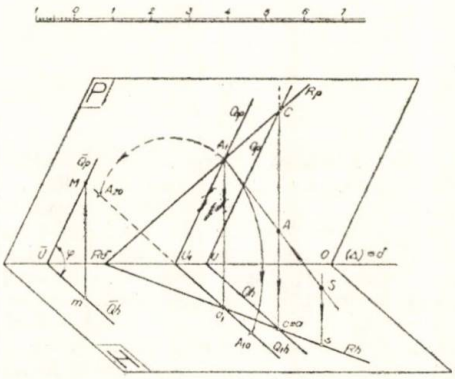

Fig. 1

Este ştiut că, construcţia imaginii (perspectivei) unui obiect cunoaşte mai multe metode printre care şi aceea denumită „dependentă" sau metodă descriptivă. Aceasta constă în a intersecta, succesiv, fiecare rază vizuală ce aparţine conului vizual, cu planul de proiecţie $[P]$, punctele de intersecţie astfel obţinute conducînd la determinarea şi deci la construcţia plană a perspectivei căutate.

În lucrarea de faţă se utilizează această metodă numai pentru determinarea omologiei afine. Adică se intersectează numai una din razele vizuale (proiectante) cu planul de proiecţie $[P]$.

2. Perspectiva punctului. Se consideră $(SA) \cap [P] = A_1$, în care (SA) este raza vizuală determinată de centrul de proiecţie S (s ; 1,5) şi de punctul A (a ; 2,5). Prin urmare punctul de intersecţie A_1 este de fapt, proiecţia centrală sau perspectiva punctului A.

Atît în schiţa din fig. 1, cît şi în epura din fig. 2, acest punct se găseşte considerînd planul $[R] = (SA) \cap (Aa)$, în care SA este proiectanta (raza vizuală) a punctului A în raport cu planul de proiecţie P, iar (Aa) este proiectanta ortogonală a aceluiaşi punct faţă de planul repre-

zentării $[H]$, deci planul $[R]$ este un plan ajutător vertical. Urma (Rh) de pe planul reprezentării este determinată de proiecţiile de pe acest plan $[H]$, a şi respectiv s ale punctelor din spaţiu A şi S. Fiind plan vertical, prin faptul că trece prin verticala (Aa), în epură, urma pe planul $[P]$ a planului $[R]$, adică dreapta (Rp) se confundă cu urma (Rh) a aceluiaşi plan. Deci $(Rh) \equiv (Rp)$.

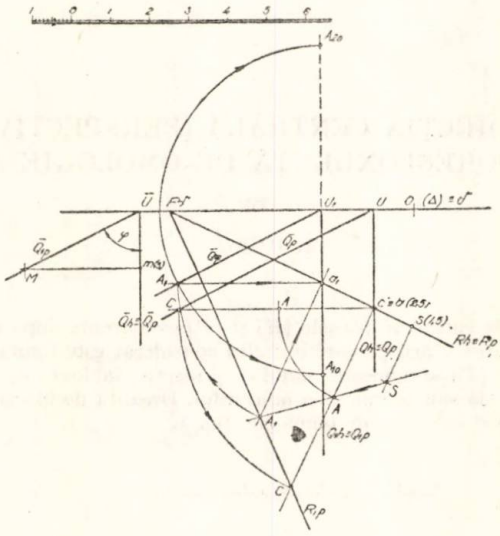

Fig. 2

Pentru construcţia urmei (Rp), ca pentru oricare dreaptă, trebuiesc cunoscute două puncte distincte ce-i aparţin. Unul din acestea este $R = (Rh) \cap (\Delta)$.

Al doilea punct utilizat este C, care rezultă din intersecţia verticalei (Aa), ce aparţine planului $[R]$ considerat, cu planul de proiecţie $[P]$. Deci $C = (Aa) \cap [P]$.

Pentru efectuarea acestei intersecţii s-a considerat planul ajutător $[Q]$ perpendicular atît pe planul $[H]$ cît şi pe planul $[P]$ şi care trece prin punctul $[A]$.

Urmele (Qh) şi (Qp) ale acestui plan sînt perpendiculare pe axa (Δ), iar urma (Qh) trece prin proiecţiile $c \equiv a$ ale punctelor C şi A situate pe aceeaşi proiectantă faţă de planul reprezentării $[H]$. Aşa cum se observă în prima figură proiectanta menţionată a punctului A este intersectată de urma (Qp) în punctul căutat C. Acest punct împreună cu urma (punctul) $R_s = (Rh) \cap (\Delta)$ determină dreapta (Rp), ce intersectează raza vizuală (SA) în A_1. Prin urmare: $A_1 = (SA) \cap [P]$, astfel încît punctul A_1 este proiecţia centrală — perspectiva — punctului A pe planul de proiecţie $[P]$ numit şi tablou. Tot pe urma (Rh) a planului vertical $[R]$ se găseşte şi proiecţia a_1 şi anume la intersecţia proiectantei punctului perspectivă A_1, faţă de planul reprezentării $[H]$, cu urma (Rh).

Trecerea la epură, la reprezentarea plană, se efectuează rabătînd planul de proiecţie $[P]$ în jurul axei (Δ), spre a se confunda cu planul reprezentării $[H]$. De aceea se consideră planul $[Q_1]$ prin proiectanta $(A_1 a_1)$, deci prin punctul A_1 şi prin proiecţia sa ortogonală a_1 de pe planul $[H]$, perpendicular pe axa (Δ) reprezentată prin urmele $(Q_1 h)$ şi $(Q_1 p)$. Efectuarea rabaterii menţionate a planului $[P]$ face ca şi punctul (perspectiva) A_1 să descrie în spaţiu un cerc cu centrul în $U_1 = [Q_1] \cap (\Delta)$ şi de rază $\overline{U_1 A_1}$. Aceasta este traiectoria rabaterii perspectivei A_1, cercul menţionat din planul ajutător $[Q_1]$ şi care intersectează urma $(Q_1 h)$ în punctele A_{10} şi A_{20}, care sînt poziţiile punctului A_1 după rabatare.

În epură, fig. 2, cele două plane $[Q]$ şi $[Q_1]$ s-au reprezentat prin urmele lor, care sînt confundate şi perpendiculare pe (Δ).

Planul $[P]$ este determinat de dreapta (Δ) şi de punctul $M(m; 3)$.

Prin punctul M s-a considerat planul $[Q]$ perpendicular de asemenea pe dreapta (Δ), deci paralel cu planele ajutătoare Q şi Q_1 şi care este apoi rabătut, împreună cu punctul M, pe planul reprezentării $[H]$. Se observă că $(Qh) \equiv (Qp)$, (Qp) ca dreaptă perpendiculară pe axa (Δ). Ducînd prin proiecţia m perpendiculara la $(Qh) \equiv (Qp)$ şi măsurînd pe aceasta $\overline{mM} = Z_M = 3$ unităţi se obţine punctul M rabătut pe planul $[H]$. Unind printr-o dreaptă punctele U şi M, se obţine urma $(Q_1 p)$, rabătută pe acelaşi plan al reprezentării $[H]$. Cu această ocazie se găseşte unghiul φ cu vîrful în U şi de laturi (Qh) şi (Qp), care este unghiul dintre planele $[H]$ şi $[P]$.

În mod analog se procedează şi cu planele $[Q]$ şi $[Q_1]$ menţionate mai sus. Astfel în epură se duc urmele (Qh), (Qp) prin proiecţia a, care intersectează axa în punctul U. Apoi prin a se duce proiectanta (perpendiculara) la $(Qh) \equiv (Qp)$ pe care se măsoară începînd din a distanţa $\overline{aA} = Z_A = 2{,}5$ unităţi. Această proiectantă aparţine şi planului vertical R pentru că $(Aa) = [Q] \cap [R]$.

De aceea tot prin a se duce perpendiculara la urma (Rh) pe care se măsoară distanţa $\overline{aA}$, în epură aceasta realizîndu-se prin trasarea unui arc de cerc cu centrul în proiecţia a şi de rază egală cu $Z_A = 2{,}5$ unităţi. Aceasta s-a făcut în ipoteza rabaterii planului $[R]$ cu elementele conţinute pe planul $[H]$.

În acelaşi fel se rabate şi centrul de proiecţie $S\,(s\,;\,1{,}5)$, astfel încît dreapta (SA) este raza vizuală a punctului A rabătută în planul $[H]$. Se observă din prima rabatere că proiectanta $(CA) \cap [Q_1 p] = C$. Punctul C, găsit astfel, este figurat şi pe perpendiculara (proiectanta) (aA), dusă în raport cu (Rh), prin trasarea arcului de cerc cu centrul în $c = a$ şi de rază $\overline{cC}$.

În acest fel $(R_\delta\, C) \equiv (R_1 p)$, care este urma planului $[R]$ pe planul de proiecţie $[P]$ rabătută în planul reprezentării $[H]$. Dreptele (SA) şi $(R_1 p)$ se intersectează în punctul A_1, perspectiva punctului A în rabatere pe planul $[H]$. Deci $A_1 = (SA) \cap (R_1 p)$.

Prin A_1 se duce proiectanta la urma (Rh), piciorul acesteia fiind punctul a_1, proiecţia ortogonală pe planul reprezentării a perspectivei A_1.

Pentru reprezentarea rabaterii în epură s-au dus prin a_1 urmele $(Q_1 h) \equiv (Q_1 p)$; ca drepte perpendiculare pe axă ele fiind urmele planului ajutător $[Q_1]$ în care este situată traiectoria circulară a rabaterii perspectivei A_1.

Tot prin proiecţia a_1 se duce acum perpendiculara la $(Q_1 h) \parallel\!\mid (Q_1 p)$, pe care se măsoară, începînd din a_1, cota $Z_A = \overline{a_1 A_1}$ şi care rezultă şi prin intersecţia perpendicularei indicate cu urma $(Q_1 p)$. Cu centrul în punctul $U_1 = [Q_1] \cap (\Delta)$ şi cu raza $\overline{U_1 A_1}$ se trasează cercul corespunzător, care se intersectează cu urma $(Q_1 h) \equiv (Q_1 p)$ în punctele A_{10} şi A_{20}. Aceste puncte reprezintă poziţiile pe care le ocupă punctul A_1 după efectuarea rabaterii planului $[P]$ spre a se confunda cu planul reprezentării $[H]$, deci sînt soluţiile rabaterii utilizată pentru trecerea la epură.

3. Construcţia perspectivei unui pătrat situat în planul reprezentării $[H]$ sau într-un plan paralel cu acesta. Fie pătratul $ABCE$ $(abce\,;\,0)$ şi punctul S $(s\,;\,2{,}5)$ centrul de proiecţie, fig. 3. Planul $[P]$ este determinat, ca şi în alte cazuri, de axa de omologie $(\Delta) \equiv (\delta)$ şi de punctul M $(m\,;\,2{,}9)$. Conul vizual se consideră proiectat ortogonal pe planul reprezentării $[H]$. De aceea se unesc prin linii drepte proiecţia s cu proiecţiile a, b, c şi e ale vîrfurilor pătratului considerat.

Pentru determinarea omologiei este necesar să se intersecteze una din razele vizuale cu planul $[P]$. În acest scop se consideră, spre exemplu raza vizuală, proiectanta, (SA) prin care se duce planul vertical $[R]$, astfel încît în epură se notează $(SA) \equiv (Rh) \equiv (Rp)$.

Pentru găsirea punctului de intersecţie se rabate planul $[R]$ pe planul $[H]$ al reprezentării, axa de rabatere fiind considerată urma (Rh). Astfel, prin proiecţia s se duce perpendiculara la (sa), pe care se măsoară, începînd din s, distanţa $\overline{sS} = Z_S = 2{,}5$ unităţi.

Planul $[R]$ se intersectează cu tabloul $[P]$ după urma (Rp), care trebuie rabătută în acelaşi plan $[H]$. Pentru aceasta se consideră orizontala $D\,(d\,;\,2{,}9)$ a punctului $M\,(m\,;\,2{,}9)$.

Cum toate punctele unei orizontale au aceeaşi cotă rezultă că şi punctul $M_1 = (D) \cap (Rp)$ va avea aceeaşi cotă, deci $Z_M = Z_{M_1} = 2{,}9$ unităţi.

Prin urmare, rabătînd planul $[R]$ pe planul $[H]$, se rabate şi urma (Rp). Pentru aceasta este suficient să se rabată punctul M_1 pe planul $[H]$, care aparţine şi urmei (Rp). Se duce, dar, perpendiculara din m_1 $(2{,}9)$ la proiecţia (sa) pe care se măsoară $\overline{m_1 M_1} = \overline{mM} = 2{,}9$ unităţi.

Unind printr-o dreaptă punctul M_1 rabătut cu punctul $U_1 = [R] \cap (\Delta)$ se obţine urma (Rp), ce reprezintă poziţia urmei (Rp) după efectuarea rabaterii. Se observă că $A_1 = (SA) \cap \cap (Rp)$, care este perspectiva vîrfului A $(a\,;\,0)$ rabătută în planul H. Din acest punct se duce perpendiculara la (sa), al cărei picior este proiecţia ortogonală a_1 pe planul $[H]$ a perspectivei A_1. Acum corespundenţa de omologie este determinată. Folosind această corespondenţă se pot găsi cu uşurinţă şi celelalte vîrfuri ale proiecţiei ortogonale ale perspectivei pătratului dat.

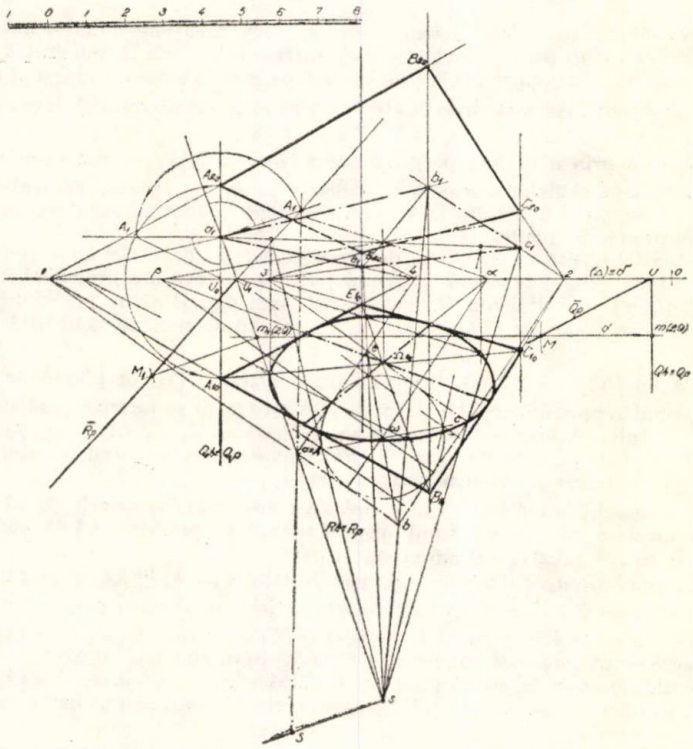

Fig. 3

În acest scop se prelungesc laturile pătratului, care intersectează axa de omologie $(\Delta) \equiv (\delta)$ în punctele *1* ; *2* ; *3* şi *4*. De exemplu, latura (ab) se intersectează cu (Δ) în punctul *1*, latura (bc) în punctul *2* şi aşa mai departe.

Unindu-se printr-o dreaptă punctele *1* şi a_1 se găseşte la intersecţia cu (sb) proiecţia b_1. De asemenea, dreapta $(2b_1)$ intersectează raza vizuală proiecţie (sc) în c_1. În mod analog se obţine vîrful e_1 în proiecţie prin intersecţia în acest punct a dreptelor $(3c_1)$; $(4a_1)$ şi (se). Deci patrulaterul $a_1\, b_1\, c_1\, e_1$ este proiecţia ortogonală pe planul reprezentării $[H]$ a perspectivei pătratului $ABCE$ considerat. Adevărata mărime a perspectivei acestui poligon se poate găsi ţinînd seama de observaţiile anterioare, prin rabaterea planului $[P]$ împreună cu perspectiva conţinută.

În acest scop se consideră prin perspectiva $A_1\,(a_1\,;\,\overline{a_1 A_1})$ planul ajutător $[Q_1]$ paralel cu planul $[Q]$ al punctului M. Se rabate acest plan $[Q_1]$ pe planul $[H]$ împreună cu punctul A_1 şi proiectanta sa pe care se măsoară cota $Z_{A_1} = \overline{a_1 A_1}$.

Deci se duce prin a_1 perpendiculara la urma $(Q_1\, h)$ ce este şi axă de rabatere, pe care se măsoară cota $Z_{A_1} = \overline{a_1 A_1}$. Aceaste se realizează trasînd arcul de cerc cu centrul în a_1 şi de

rază $Z_{A_1} = \overline{a_1 A_1}$. Apoi cu centrul în punctul $U_0 = [Q_1] \cap (\Delta)$ şi de rază $\overline{U_0 A_1}$ se trasează un cerc, care intersectează urma $(Q_1 h) \equiv (Q_1 p)$ în punctul A_{10} şi respectiv A_{20}. Acestea sînt soluţiile rabaterii perspectivei A_1 a vîrfului A pe planul reprezentării $[H]$.

Se remarcă faptul că rabaterea planului $[P]$, aşa cum s-a arătat, face ca vîrfurile A_1; B_1; C_1 şi E_1, ale perspectivei conţinute, să descrie în spaţiu cercuri situate în plane perpendiculare pe axa de omologie $(\Delta) \equiv (\delta)$. De aceea, prin a_1 (aşa cum s-a arătat) şi prin b_1; c_1 şi e_1 se duc perpendiculare pe (Δ), care se intersectează cu dreptele afine astfel. Dreapta $(1A_{10})$ se intersectează cu perpendiculara la (Δ) ce trece prin b_1 în B_{10}, iar dreapta $(1\,A_{20})$ se intersectează cu aceeaşi perpendiculară în punctele B_{20} ş.a.m.d.

Patrulaterele $A_{10}\,B_{10}\,C_{10}\,E_{10}$ şi $A_{20}\,B_{20}\,C_{20}\,E_{20}$ reprezintă perspectiva reală a pătratului $ABCE$ dat pe tabloul $[P]$ în cele două soluţii.

Construcţia perspectivei unui pătrat conduce într-un mod simplu şi la construcţia perspectivei cercului înscris, care este o conică. Astfel s-a considerat cercul ω înscris în pătratul proiecţiei *abce*, în care s-au figurat diagonalele şi apotemele. Extremităţile apotemelor sînt punctele în care laturile pătratului sînt tangente la cercul ω. De asemenea, acelaşi cerc intersectează diagonalele pătratului circumscris încă în patru puncte. Cele opt puncte indicate se iau în considerare la construcţia elipsei ca perspectivă a cercului.

Astfel, apotemele prelungite întîlnesc axa de omologie în punctele α şi β. Aceste puncte unite prin drepte cu intersecţia ω_{10} a diagonalelor din perspectiva $A_{10}\,B_{10}\,A_{10}\,E_{10}$ conduc la găsirea punctelor de intersecţie cu laturile perspectivei menţionate. Prin aceste puncte trece elipsa $\Omega\,(\omega\,;0)$, care este tangentă la patrulaterul perspectivă $A_{10}\,B_{10}\,C_{10}\,E_{10}$. Considerînd razele vizuale prin punctele în care diagonala părtatului intersectează cercul înscris, acestea se intersectează cu diagonalele în paralelogramul proiecţie $a_1\,b_1\,c_1\,e_1$ în punctele prin care ducîndu-se linii de ordine în raport cu (Δ) se găsesc pe diagonalele din perspectiva menţionată (perspectivele acestor puncte). Se pot utiliza şi liniile afine ca drepte ale acestor puncte. Prin cele opt puncte perspectivă găsite se poate trasa elipsa Ω_{10}, ca perspectivă a cercului ω $(\Omega\,;0)$.

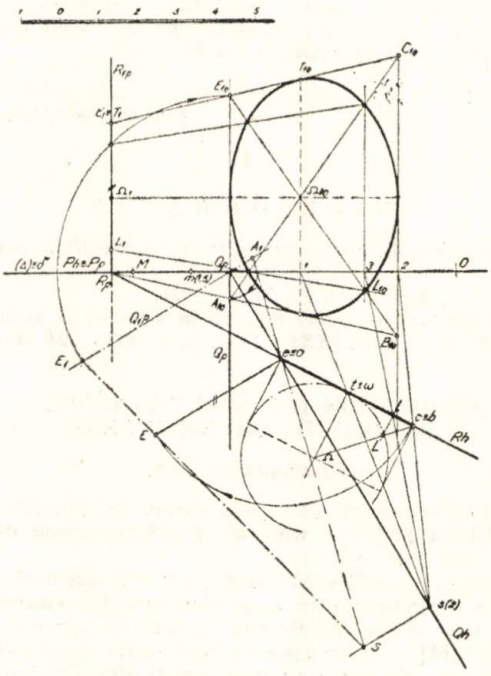

Fig. 4

4. Perspectiva figurilor geometrice situate în plane verticale. Se consideră acelaşi sistem de referinţă format din planul $[H]$ al reprezentării şi planul de proiecţie $[P]$ un plan vertical, care se intersectează cu primul după dreapta $(\Delta) = [H] \cap [P]$. În fig. 4, aceasta trece prin punctul M $(m\,;\,1{,}5)$.

S-a considerat pătratul $ABCE$ şi cercul Ω înscris, ambele figuri geometrice fiind situate în planul vertical $[R]$. Aşa fiind proiecţia pe planul $[H]$ al reprezentării a figurilor plane date se face pe urma (Rh) a planului ce le conţine, proiecţie notată cu $abce\omega tf$. Primele patru puncte proiecţii reprezintă vîrfurile pătratului considerat, proiecţia ω reprezintă centrul cercului înscris, t este proiecţia punctelor în care cercul înscris este tangent la laturile pătratului, iar l reprezintă proiecţia punctelor în care cercul ω este intersectat de diagonalele pătratului $ABCE$.

Din figură se desprinde faptul că latura AB este o orizontală de cotă zero şi este proiectată pe urma (Rh). Prin urmare, două din laturile pătratului sînt orizontale şi celelalte două sînt verticale.

Fie centrul de proiecţie $S\,(s\,;\,2)$. Acext punct şi cu dreptele verticale (laturi, coarde etc.) ale figurilor plane date, determină planele vizuale $[Q]$, care se intersectează cu tabloul vertical $[P]$ după verticalele (Qp), ce reprezintă locul perspectivelor (proiecţiilor centrale) ale dreptelor indicate. Pe aceste drepte se găsesc proiecţiile centrale A_{10} ; B_{10} ; C_{10} ; E_{10} ; I_{10} ; T_{10} şi L_{10}, care determină perspectiva celor două figuri plane date.

Se consideră planul vertical $[Q]$ determinat de razele vizuale (SA) şi (SE). Acest plan vertical se intersectează cu tabloul după dreapta $(Qp) = [Q] \cap [P]$.

Se rabate planul $[Q]$ în planul reprezentării, axa de rabatere fiind urma (Qh), împreună cu dreapta $(Qp) \rightarrow (Q_1 p)$ şi razele vizuale (SA) şi (SE). Intersecţia acestora cu $(Q_1\,p)$ conduce la găsirea punctelor A_1 şi E_1, care sînt proiecţiile centrale (perspectivele) punctelor A şi respectiv E rabătute în planul reprezentării. Aplicînd metoda ridicării, aceste puncte ajung pe urma (Qp) în poziţiile A_{10} şi E_{10}, care reprezintă perspectiva pe tabloul $[P]$ a vîrfurilor menţionate.

Celelalte puncte caracteristice figurilor plane date se găsesc considerînd corespondenţa de omologie afină dintre figurile date (pătratul şi cercul înscris) şi proiecţiile centrale ca şi figurile conţinute, urma (Rp) fiind axa lor de omologie.

5. Concluzii. Prin folosirea corespondenţei de omologie afine, perspectiva elementelor şi figurilor geometrice plane se construieşte cu uşurinţă şi cu un grad de precizie corespunzător. Atît metoda indicată în acest sens, cît şi diedrul de referinţă variabil adoptate reprezintă elementul original, util în proiectare, cercetare şi învăţămlnt.

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Institutul politehnic, Iaşi,
Catedra de geometrie descriptivă şi desen

BIBLIOGRAFIE

1. Gheorghiu A., *Tehnica desenului perspectiv în construcţii şi arhitectură.* E.T.B., Buc., 1963.
2. Tănăsescu A., *Perspectiva.* E.D.P., Buc., 1971.
3. Rusu I., *Asupra adaptării proiecţiei centrale în sistemul de referinţă plan-axial.* Bul. Instit. polit. Iaşi, XVII (XXI), *1—2*, s. I, 117—124 (1971).

MITTELPROJEKTION (PERSPEKTIV) DURCH AFFINE HOMOLOGIEKORRESPONDENZ

(Zusammenfassung)

Die Arbeit legt die Perspektivkonstruktion ebener geometrischer Elemente und Figuren am $[P]$-Bild dar. Dieses $[P]$-Bild wird als Projektionsebene des als Referenzsystem verwendeten variablen Dieders betrachtet.

Auf der Darstellungsebene $[H]$ wird das geometrische Element durch seine senkrechte Projektion gegeben. Die Projectionsmitte zusammen mit den einzelnen den vorgegebenen geometrischen Element oder Figur charakterisierenden Raumpunkte, bestimmt den Sichtkegel. Dessen Schnitt mit Bild $[P]$ ist die gesuchte Perspektive, welche man auf Grund der affinen Homologiekorrespondenz die zwischen den beiden Projektionen besteht, konstruieren kann.

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D. C. 530.01

ON THE ARITHMETIC BASIS OF SPECIAL RELATIVITY

BY

DONALD GREENSPAN

1. Introduction. It is known [1]—[3] that in special relativity many of the concepts can be formulated and many of the results deduced using only arithmetic processes, and, in particular, differences. The complete arithmetization has not been achieved because of the absence of an arithmetic force law which not only converges to the continuous force law in the limit, but is also invariant under the Lorentz transformation. Our purpose is to produce such an arithmetic formula.

2. Basics. Consider two Euclidean coordinate systems XYZ and $X'Y'Z'$ which, at some initial time, coincide. Let the $X'Y'Z'$ system, called the rocket frame, be in constant uniform motion relative to the XYZ frame, called the lab frame. Assume the constant relative speed is u and that axes Y and Y' are always parallel, as are axes Z and Z'. For $\Delta t > 0$, an observer in the lab frame makes observations at the distinct times $t_k = k\Delta t$, $k = 0, 1, 2,...$ Using an identical clock, an observer in the rocket frame makes observations at the times t'_k, $k = 0, 1, 2,...$, where t'_k on the rocket clock corresponds to t_k on the lab clock. Now, if particle P is at (x_k, y_k, z_k) in the lab frame at time t_k, while it is at (x'_k, y'_k, z'_k) in the rocket frame at time t'_k, then the events (x_k, y_k, z_k, t_k) and (x'_k, y'_k, z'_k, t'_k) are related by the Lorentz transformation

$$x'_k = \frac{c\,(x_k - ut_k)}{(c^2 - u^2)^{1/2}}, \quad \text{or} \quad x_k = \frac{c\,(x'_k + ut'_k)}{(c^2 - u^2)^{1/2}}, \tag{2.1}$$

$$y'_k = y_k, \tag{2.2}$$

$$z'_k = z_k, \tag{2.3}$$

$$t'_k = \frac{c^2 t_k - ux_k}{c\,(c^2 - u^2)^{1/2}}, \quad \text{or,} \quad t_k = \frac{c^2 t'_k + ux'_k}{c\,(c^2 - u^2)^{1/2}}, \tag{2.4}$$

where c is the speed of light.

In the lab frame, let particle P be in motion in the X-direction. Then at time t_k, P' s velocity $v(t_k) = v_k$ and acceleration $a(t_k) = a_k$ are defined by

$$v_k = \frac{\Delta x_k}{\Delta t_k}, \tag{2.5}$$

$$a_k = \frac{\Delta v_k}{\Delta t_k}, \tag{2.6}$$

where Δ is the usual forward difference operator defined by

$$\Delta F(k) = F(k+1) - F(k). \tag{2.7}$$

Similarly, in the rocket frame, at time t'_k one defines v'_k and a'_k by

$$v'_c = \frac{\Delta x'_k}{\Delta t'_k}, \tag{2.8}$$

$$a'_k = \frac{\Delta v'_k}{\Delta t'_k}. \tag{2.9}$$

In order to find the relationships between v_k and v'_k, and between a_k and a'_k, note first that the linearity of (2.1)—(2.4) implies

$$\Delta x'_k = \frac{c\,(\Delta x_k - u\,\Delta t_k)}{(c^2 - u^2)^{1/2}}, \tag{2.10}$$

$$\Delta y'_k = \Delta y_k, \tag{2.11}$$

$$\Delta z'_k = \Delta z_k, \tag{2.12}$$

$$\Delta t'_k = \frac{c^2\,\Delta t_k - u\,\Delta x_k}{c\,(c^2 - u^2)^{1/2}}. \tag{2.13}$$

Thus, (2.8), (2.10) and (2.13) imply

$$v'_k = \frac{c^2\,(v_k - u)}{c^2 - u v_k}, \tag{2.14}$$

while (2.9), (2.13) and (2.14) imply

$$a'_k = \frac{c^3\,(c^2 - u^2)^{3/2}}{(c^2 - u v_k)^2\,(c^2 - u v_{k+1})}\, a_k. \tag{2.15}$$

Assume also that particle P has mass m at time t_k in the lab frame and mass m' at the corresponding time t'_k in the rocket frame, and that

$$m = \frac{c m_0}{(c^2 - v_k^2)^{1/2}}, \tag{2.16}$$

$$m' = \frac{c m_0}{(c^2 - v_k'^2)^{1/2}}, \tag{2.17}$$

where m_0 is a constant called the rest mass of P.

3. An Invariant Arithmetic Force Formula. From the dynamical point of view, the actual motion of a particle in, say, the lab frame can be determined from (2.5) and (2.6) once an equation which relates force and acceleration is given. We now take this equation to be

$$F_k = \frac{c^2 m}{[(c^2 - v_k^2)(c^2 - v_{k+1}^2)]^{1/2}} \frac{\Delta v_k}{\Delta t_k} . \tag{3.1}$$

By the principle of relativity, however, we must have in the rocket frame

$$F'_k = \frac{c^2 m'}{[(c^2 - v_k'^2)(c^2 - v_{k+1}'^2)]^{1/2}} \frac{\Delta v'_k}{\Delta t'_k} . \tag{3.2}$$

But this is valid only if the right-hand side of (3.2) transforms into the right-hand side of (3.1) under the Lorentz transformation. This is, indeed, the case, since

$$\frac{c^2 m'}{[(c^2 - v_k'^2)(c^2 - v_{k+1}'^2)]^{1/2}} \frac{\Delta v'_k}{\Delta t'_k} = \frac{c^3 m_0}{(c^2 - v_k'^2)(c^2 - v_{k+1}'^2)^{1/2}} a'_k =$$

$$= \frac{c^3 m_0}{(c^2 - v_k^2)(c^2 - v_{k+1}^2)^{1/2}} = \frac{c^2 m}{[(c^2 - v_k^2)(c^2 - v_{k+1}^2)]^{1/2}} \frac{\Delta v_k}{\Delta t_k} .$$

Note that taking limits in (3.1) yields the particular form

$$F = \frac{c^2 m}{c^2 - v^2} \frac{\mathrm{d}v}{\mathrm{d}t} \tag{3.3}$$

of the classical Einstein equation

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(mv) , \tag{3.4}$$

where

$$m = \frac{cm_0}{(c^2 - v^2)^{1/2}} .$$

Finally, it should be noted that arithmetic formulas (3.1) and (3.2) enable one to use identical modern computers in both the lab and rocket frames with the assurance that the resulting numerical calculations will be Lorentz related.

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University of Wisconsin,
Madison, Wisconsin, USA

REFERENCES

1. Taylor E. F., Wheeler J. A., *Space-Time Physics*. Freeman, San Francisco, 1966.
2. Greenspan D., *A Finite Difference Proof that $E = mc^2$*. Amer. Math. Mo., **80**, 289–292 (1973).
3. Greenspan D., *The Arithmetic Basis of Special Relativity*. Tech. Rpt. *232*, Dept. Comp. Sci., Univ. Wis., Madison, Dec., 1974.

ASUPRA BAZELOR ARITMETICE ALE RELATIVITĂŢII RESTRÎNSE

(Rezumat)

Se deduc formulele aritmetice (3.1), (3.2) pentru exprimarea forţei în sisteme de referinţă aflate în mişcare relativă rectilinie şi uniformă, care nu numai converg la limită către forţa continuă, dar sînt şi invariante în raport cu transformarea Lorentz.

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D.C. 539.31

A METHOD OF SUCCESIVE APPROXIMATION IN NONLINEAR THEORY OF SIMPLE DIPOLAR ELASTIC SOLIDS (II)

BY

SILVIA G. NEAGU

This paper is a continuation of [1]. Here we present a procedure for the calculation of the displacement field and of the microdeformation field corresponding to the equations of n-th order elasticity theory for dipolar solids, following Rivlin and Topakoglu [2], [3].

We assume that W, which was defined in [1] is an analytic function of $U_{K,L}$, Φ_{KL}, $\Phi_{KL,M}$. Let us introduced the notations $H = (U_{K,L})$, $\Phi = (\Phi_{KL})$, $\Phi_M = (\Phi_{KL,M})$. Thus we can write $W = W(H, \Phi, \Phi_M)$ and

$$W = \sum_{n=1}^{\infty} W_n, \tag{3.1}$$

where W_n is a homogeneous polynomial of degree n in H, Φ and Φ_M.

Since the components T_{Ki} $\bar{\sigma}_{,Ki}$, μ_{Mij} are given by [1, (1.8)—(1.9)] they are also analytical functions of H, Φ, Φ_M.

Let us denote

$$\begin{cases} T = t(H, \Phi, \Phi_M), \\ \bar{\sigma} = Z(H, \Phi, \Phi_M), \\ \mu = \eta(H, \Phi, \Phi_M). \end{cases} \tag{3.2}$$

Obviously we can write

$$\begin{cases} t(H, \Phi, \Phi_M) = \sum_{s=1}^{\infty} t_s(H, \Phi, \Phi_M), \\ Z(H, \Phi, \Phi_M) = \sum_{s=1}^{\infty} Z_s(H, \Phi, \Phi_M), \\ \eta(H, \Phi, \Phi_M) = \sum_{s=1}^{\infty} \eta_s(H, \Phi, \Phi_M), \end{cases} \tag{3.3}$$

where t_s, Z_s, η_s are homogeneous polynomials of degree n in H, Φ and Φ_M.

The equations of motion and boundary conditions for the deformation of the dipolar elastic body are given by [1, (1.11)—(1.12)]. If the deformation is such that $U_{i,K}$, Φ_{KL}, $\Phi_{KL,M}$ are sufficiently small compared with unity, it can approximate to W by the expression W_1.

To a higher degree of approximation we can take $W = W_2 + W_3$ and so on. Let us denote the partial sums of the expansions [1, (2.1)—(2.2)], (3.2), (3.3) by T_1^n, T_2^n, f^n, F^n, U^n, Φ^n, H^n, Φ_M^n, T^n, $\overline{\sigma}^n$, μ^n, t^n, Z^n and η^n respectively. For example

$$t^n(H, \Phi, \Phi_M) = \sum_{s=1}^{n} t_s(H, \Phi, \Phi_M), \qquad f^n = \sum_{r=1}^{n} \varepsilon^r f^{(r)}. \tag{3.4}$$

From (3.3) and (2.6) we have

$$t^n(H^n, \Phi^n, \Phi_M^n) = \sum_{s=1}^{n} t_s\left[\sum_{r=1}^{n} \varepsilon^r H^{(r)}, \sum_{r=1}^{n} \varepsilon^r \Phi^{(r)}, \sum_{r=1}^{n} \varepsilon^r \Phi_M^{(r)}\right] =$$

$$= T^n + O(\varepsilon^{n+1}) = \sum_{r=1}^{n} \varepsilon^r [t_1(H^{(r)}, \Phi^{(r)}, \Phi_M^{(r)}) + N^{(r)}(H^{(1)}, \ldots, H^{(r-1)},$$

$$\Phi^{(1)}, \ldots, \Phi^{(r-1)}, \Phi^{(1)}, \ldots, \Phi_M^{(r-1)})] + O(\varepsilon^{n+1}),$$

$$Z^n(H^n, \Phi^n, \Phi_M^n) = \overline{\sigma}^n + O(\varepsilon^{n+1}) = \sum_{r=1}^{n} \varepsilon^r [Z_1(H^{(r)}, \Phi^{(r)}, \Phi_M^{(r)}) + \tag{3.5}$$

$$+ S^{(r)}(H^{(1)}, \ldots, H^{(r-1)}, \Phi^{(1)}, \ldots, \Phi^{(r-1)}, \Phi_M^{(1)}, \ldots, \Phi_M^{(r-1)})] + O(\varepsilon^{n+1}),$$

$$\eta^n(H^n, \Phi^n, \Phi_M^n) = \mu^n + O(\varepsilon^{n+1}) = \sum_{r=1}^{n} \varepsilon^r [\eta_1(H^{(r)}, \Phi^{(r)}, \Phi_M^{(r)}) +$$

$$+ R^{(r)}(H^{(1)}, \ldots, H^{(r-1)}, \Phi^{(1)}, \ldots, \Phi^{(r-1)}, \Phi_M^{(1)}, \ldots, \Phi_M^{(r-1)})] + O(\varepsilon^{n+1}),$$

where $O(\varepsilon^{n+1})$ indicates terms of order ε^{n+1} and superior to $n+1$. From (2.7) it follows that

$$\begin{cases} \operatorname{Div} T^n + \operatorname{Div} \overline{\sigma}^n + \rho_0 f^n = 0, \\ \overline{\sigma}^n = \rho_0 F^n + \operatorname{Div} \mu^n - (\overline{\sigma} H)^{n-1}, \\ (T^n + \overline{\sigma}^n) N = T_1^n, \qquad \mu^n N = T_2^n. \end{cases} \tag{3.6}$$

So, it follows from (3.5)

$$\begin{cases} \operatorname{Div} t^n(H^n, \Phi^n, \Phi_M^n) + \operatorname{Div} Z^n(H^n, \Phi^n, \Phi_M^n) + \rho_0 f^n = O(\varepsilon^{n+1}), \\ Z^n(H^n, \Phi^n, \Phi_M^n) = \rho_0 F^n + \operatorname{Div} \eta^n(H^n, \Phi^n, \Phi_M^n) - (ZH)^{n-1} + O(\varepsilon^{n+1}), \\ [t^n(H^n, \Phi^n, \Phi_M^n) + Z^n(H^n, \Phi^n, \Phi_M^n)] N = T_1^n + O(\varepsilon^{n+1}), \\ \eta^n(H^n, \Phi^n, \Phi_M^n) N = T_2^n + O(\varepsilon^{n+1}). \end{cases} \tag{3.7}$$

These equations may be called the equations of equilibrium and boundary conditions of n-th order elasticity theory for dipolar solid. We

shall say we have solved the equations of n-th order elasticity for dipolar solid if we have found a polynomial solution

$$U^n = \sum_{r=1}^{n} \varepsilon^r U^{(r)}, \qquad \Phi^n = \sum_{r=1}^{n} \varepsilon^r \Phi^{(r)}. \tag{3.8}$$

Now, we will give the procedure for obtaining a solution of the equations (3.7). We suppose that we have a solution U^{n-1}, Φ^{n-1} of the equations of $n-1$-th order in the above defined sense. We introduce U^{n-1}, Φ^{n-1} in the equations (3.7) of n-th order and calculate the body and dipolar body forces and surface and dipolar surface tractions corresponding to the solution U^{n-1}, Φ^{n-1} according to the equations of equilibrium and boundary conditions of n-th order in the theory of dipolar elastic solid. Hence, we obtain

$$\begin{gathered}
\operatorname{Div} t^n [H^{n-1}, \Phi^{n-1}, \Phi_M^{n-1}] + \operatorname{Div} Z^n [H^{n-1}, \Phi^{n-1}, \Phi_M^{n-1}] + \\
+ \rho_0 (f^n - \varepsilon^n \overset{*}{f}{}^{(n)}) = O(\varepsilon^{n+1}), \\
Z^n [H^{n-1}, \Phi^{n-1}. \Phi_M^{n-1}] = \rho_0 (F^n - \varepsilon^n \overset{*}{F}{}^{(n)}) + \operatorname{Div} \eta^n [H^{n-1}, \Phi^{n-1}, \Phi_M^{n-1}] - \\
- (ZH)^{n-2} + O(\varepsilon^{n+1}), \\
[t^n (H^{n-1}, \Phi^{n-1}, \Phi_M^{n-1}) + Z^n (H^{n-1}, \Phi^{n-1}, \Phi_M^{n-1})] N = \\
= T_1^n - \varepsilon^n \overset{*}{T}{}^{(n)} + O(\varepsilon^{n+1}), \\
\eta^n (H^{n-1}, \Phi^{n-1}, \Phi_M^{n-1}) N = T_2^n - \varepsilon^n \overset{**}{T}{}^{(n)} + O(\varepsilon^{n+1}),
\end{gathered} \tag{3.9}$$

where $-\varepsilon^n \overset{*}{T}{}^{(n)}$, $-\varepsilon^n \overset{**}{T}{}^{(n)}$, $-\varepsilon^n \overset{*}{f}{}^{(n)}$, $-\varepsilon^n \overset{*}{F}{}^{(n)}$ are given by [1, (2.9)] and represent the surface and dipolar surface and the body and dipolar body force, which would be required in addition, in order to maintain the displacement U^{n-1} and microdeformation Φ^{n-1} in equilibrium according to the order theory.

Equations [1, (2.8)] show that $\varepsilon^n U^{(n)}$, $\varepsilon^n \Phi^{(n)}$ represent a solution in the infinitesimal theory corresponding to the loads $\varepsilon^n \overset{*}{T}{}^{(n)}$, $\varepsilon^n \overset{**}{T}{}^{(n)}$, $\varepsilon^n \overset{*}{f}{}^{(n)}$, $\varepsilon^n \overset{*}{F}{}^{(n)}$.

Then we can show that $U^n = U^{n-1} + \varepsilon^n U^{(n)}$, $\Phi^n = \Phi^{n-1} + \varepsilon^n \Phi^{(n)}$ satisfies the equations of equilibrium and boundary conditions of n-th order theory.

Consequently, by repeating succesively the procedure described with n equal to $1, 2, 3, \ldots, n-2$ we can calculate the displacement field U^{n-1} and the microdeformation field Φ^{n-1} corresponding to the equations of $n-1$-th order elasticity theory for dipolar solid.

Then the partial sums U^n, Φ^n of expansion [1, (2.2)] gives a solution of the equations of n-th order and every solution is of this form.

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Polytechnic Institute, Jassy
Department of Mechanics

REFERENCES

1. Neagu G. S., *A Method of Successive Approximation in Nonlinear Theory of Simple Dipolar Elastic Solid* (I). Bul. Inst. polit. Iaşi, XXII (XXVI), *3–4*, s. I (1976)
2. Rivlin R. S., Topakoglu C., *A Theorem in the Theory of finite Elastic Deformations*. J. of Rat. Mech. and Analysis, **3**, 581–589 (1954).
3. Truesdell C., Noll W., *The Nonlinear Field Theories of Mechanics*. Handbuch der Physik, III, *3*, Springer Verlag, Berlin, 1965.

O METODĂ DE APROXIMAŢII SUCCESIVE ÎN TEORIA NELINIARĂ A SOLIDULUI ELASTIC DIPOLAR SIMPLU (II)

(Rezumat)

Lucrarea, care este o continuare a lui [1], prezintă un procedeu pentru calcularea soluţiei corespunzînd ecuaţiilor de ordinul n în teoria elasticităţii pentru solide dipolare, urmînd pe Rivlin şi Topakoglu [2], [3].

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D. C. 621.319.2

TSC SPECTRUM OBTAINED FROM ELECTRON BEAM CHARGED PET FILM

BY

ALAJDIN ABAZI

1. Introduction. One of the methods of charging a polymer film e.g. to produce an electret is to bombard with electron beams. As is shown the advantages of electron beam charging is that by scaning the beam over the surface a very uniform charge distribution can be obtained, and that charges be deposited at a known depth. In this paper samples of 23 μm PET film bombarded with a 5 kV, 10^{-7} A beam current for $\sim$ 3 minutes in the O.C. configuration are presented. A typical T.S.C. spectrum is obtained and 3 peaks are apparent.

As is known the electron beam is a very powerful diagnostic tool in materials analysis. One needs only to consider this first diagram (Fig. 1).

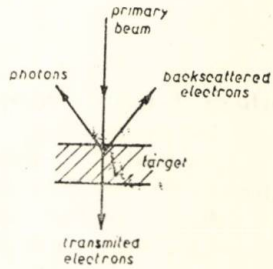

Fig. 1

Transmitted electrons — transmission microscopy, energy loss microscopy, Backscattered electrons — scaning electron microscopy ; photons X-ray microprobe analysis, cathodoluminescence. In the bulk electron beam produces hole electron pairs by ionisation and so can conduct time of flight measurements to determine carrier mobility.

In addition to all this, if the target material is relatively insulating, e.g. a polymer, then the rate of accumulation of charge can exceed the rate of dissipation and so the target charges up. After terminating the bombardment polymeric materials retain this excess charge for significant periods of time. It is this ability to retain charge that makes polymers such good electret materials.

One of the standard methods of charging, a polymer to produce an electret is to expose one surface to a corona discharge. However, this leads to a fairly non-uniform charge distribution over the surface.

One of the advantages of electron beam charging is that by scanning the beam over the surface a very uniform charge distribution can de obtained even over areas of 5×5 cm^2 (fig. 2).

Another advantage of the electron beam is that charge can be deposited at a known depth below the sample surface. The range of an electron beam in a material follows an empirical law of the form

$$X_R = AE_p^n, \tag{1}$$

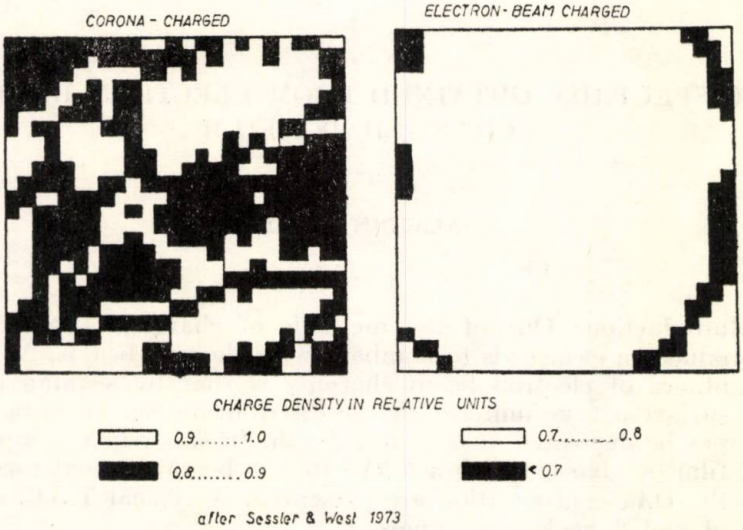

Fig. 2

where A and n depend on the target material. In general $n=1.3$ for metals and 1.7 for polymers. Although A and n can be determined experimentally they can also be determined theoretically using the energy loss relation developed by Bethe, Rose and Smith (Fig. 3). Their calculation was based on the complete diffusion model.

Electrons are assumed to travel in straight lines to a depth X_d—depth of complete diffusion. At this depth an electron undergoing a collision has an egual probability of being scattered in any direction. The scattered electron diffuses to the outer circumference of the circle radius $X_R - X_d$. To find X_R integrate from E_P to 0. Rule of thumb for organics need a few kV per μm of penetration. For efficient charging need to choose a beam energy that ensures that the range is less than sample thickness. Furthermore from the mode operation of an electret is important to ensure that the charge is located nearer the upper surface than the lower surface (ensures larges external field for given electrode spacing).

The sample can be sandwiched between evaporated metal electrodes (the upper being almost transparent to the beam) short-circuited to earth. We have included two electrometers in the circuit (Fig. 4) because it is instructive to monitor the current flow in the circuit during the bombardment.

A beam current I is deposited in the sample (strictly $I_0 = I_{\text{beam}} - I_{\text{backscattered}}$). When the beam is first switched on $I_1 = 0$ and $I_2 = 0$ and is entirely a displacement current. But very quickly as the space-charge grows, a field builds up causing a leakage current I_1 to flow to the front electrode through the more conductive irradiated layer. This causes I_2 to diminish. As time progresses I_1 builds up $\sim I_0$ while I_2 decreases $\sim 10^{-3}$... ... $10^{-5} I_0$ and represents the leakage through the unirradiated region. The time to reach saturation is probably a few seconds.

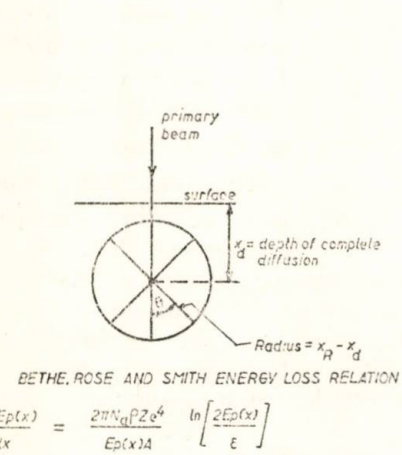

Fig. 3

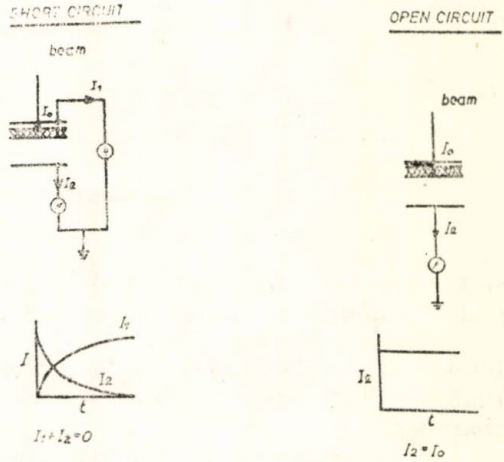

Fig. 4

The second option is to bombard with the front electrode disconnected from earth and allowed to float. In this situation $I_2 \simeq I_0$ and mesures the rate of interception of charge by the front surface. Obviously the potential of the front surface will rise negatively. This increasing negative potential will probably cause a decrease in penetration depth, and cause an increase in secondary emission which then will probably decrease I_0. So in practice I_2 will show a slow decrease. Increased emission and increased leakage through the sample will ultimately limit the amount of charge trapped. This limit is probably reached over a period of a few minutes.

So immediately we can infer that more is likely to be injected into the polymer using the open-circuit charging procedure.

Gross, Sessler and West have also shown that by a simple modification the O.C. charging method can lead to a method for determining the approximate depth of the charge layer.

The sample is charged in O.C. with capacitor C_2 to replace the electrometer. During bombardment q_2 builds up linearly. If the bombardment is terminated and the switch closed then due to field rearrangements in the sample a displacement charge q_1 appears on C_1 and q_2 decreases. This is due to the field rearrangements within the sample after short circuiting.

From their equation found $d \simeq 2R$.

So it is comforting to know that the charge is centered somewhere near the practical range of the beam (Fig. 5).

The charge lifetime in electron bombarded samples have been determined by Sessler and West and have been found to be as long if not longer than from corona charged samples. In fact some polymers have shown a projected lifetime in excess of 200 years. The exact nature of the trapping sites is unclear and many workers including Van Turnhout, Creswell and Perlman, Sessler and West have attempted to use the thermally stimulated current (TSC) method to analyse the trapping process.

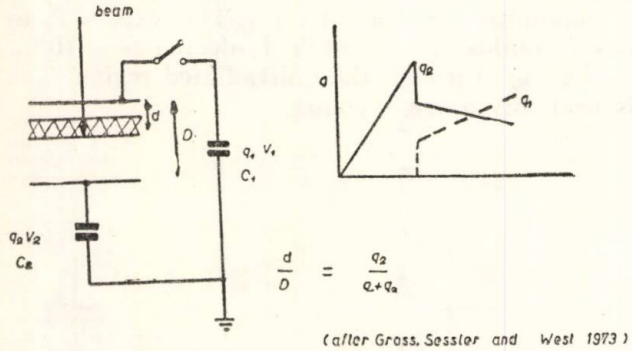

Fig. 5

2. Experimental Work. In the present work samples bombarded in both S.C. and O.C. configuration were studied but it was soon apparent that the TSC from S.C. samples was far smaller than from the O. C. charging.

Samples of 23 μm PET film were charged with a 5 kV, 10^{-7} A beam current for 3 minutes. The range 0.8 μm. The beam was defocussed to a spot size ~ 1 cm² (Fig. 6). After bombarding, the contact to the front electrode was remade and the sample heated and the current from the sample monitored as a function of temperature. The heating rate was of the order of 2° C/min.

A typical TSC spectrum is shown from two different samples and gives some idea of the reproductibility (Fig. 7).

Several points worthy of mention.

1° The current flow during discharge is consistent with the flow of electrons back to the front electrode.

2° 3 peaks are apparent. Using the initial rise and peak cleaning the first two peaks found to be 0.4 ± 0.05 eV and 0.8 ± 0.05 eV. The third peak has not been entirely isolated yet but seems to be 1.5 eV.

The existence of these levels in PET has been confirmed by several workers, e.g. from TSC of corona charged PET Creswell and Perlman found levels at 0.55, 0.8 and 1.4 eV, Lilly at all from the TSC of PET charged with high fields found traps at 0.5—0,6, 0.7—0.8, 1.4 eV. While Hirch in his studies of electron mobility in PET found a mobility activation energy rising to 0.35 eV as the sample temperature increased to 100° C.

3° Since we believe that the TSC peaks are due to electron activation release from the irradiated area to the front electrode then an estimate of the total charge trapped can be made.

This is done by estimating the total charge released during the TSC and then multiplying by D/d. Where D = thickness, d = range.

It is found that some 10% of the intercepted charge was released during the TSC. This may mean that more charge remains in the sample presumably in very deep traps or, as is more likely, only some 10% of the primary beam current was trapped in the sample. From the charge released however it is estimated that the trap density in the present samples of PET is $\sim 10^{17}$ cm^{-3}.

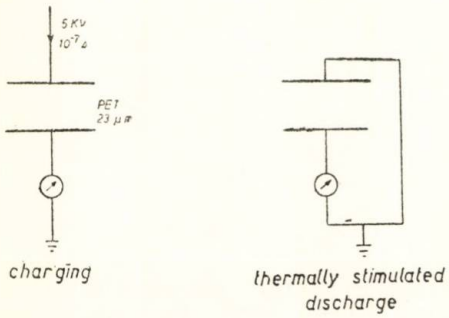

Fig. 6

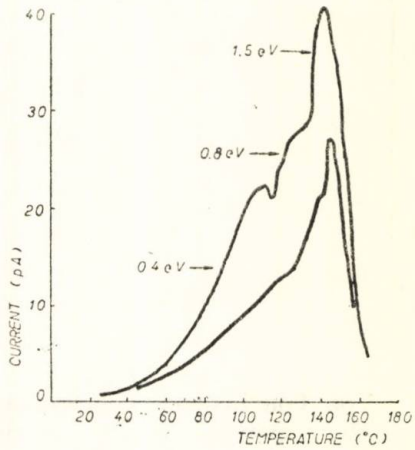

Fig. 7

This is in good agreement with the figures of $10^{17}-10^{20}$ predicted by Fowler from his measurements of X ray induced conductivity.

If the total radiation dose during bombarding is increased for $\sim 10^5$ Mrads it is found that the discharge current is reversed.

However, we could not rise the temperature sufficiently high to enable the TSC peak to be observed. However, the discharge current behaved in all other respects like a charge realese, e.g. if the temperature held at 175° C the current decreases with time and if the temperature cycle was repeated the current remaines small. From the initial rise the trap depth would appear to be 1.65 ± 0.1 eV (Fig. 8).

We believe that the explanation for this is as follows. Because of the high radiation dose radiation damage occurs in the formation of deep traps thus rendering the intercepted charge immobile in the time scale and temperature range of the experiment. This is in fact a model proposed by Caleman and Bohm in 1953 to account for a permanent decrease in radiation induced conductivity after a small initial rise.

The current flow during the geating cycle is then seen to be a neutralisation of the trapped charge by the flow of thermally generated carriers (or dipole orientation) in the bulk of the sample and so the assumed trap depth of 1.65 eV refers to the activation energy for conduction in the PET (Fig. 9). In this connection it is interesting that Reddish many years ago reported that a low frequency process with an activation energy of

1.6 eV occurred in PET. He attributed this to a d.c. conduction process. This activation energy may also be related to the 1.5 eV trap found in the previous slide. Finally the current flow up to 175° C represents a neutralisation of only 10% of the trapped charge, and in part may explain why radiation hardening of polymers leads to better charge stability.

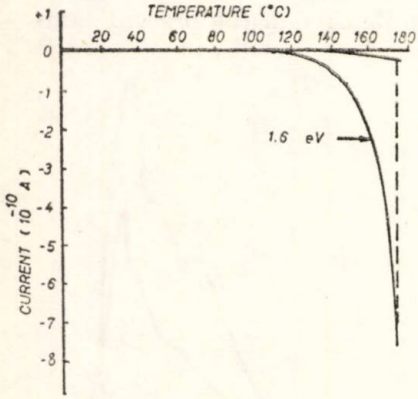

Fig. 8

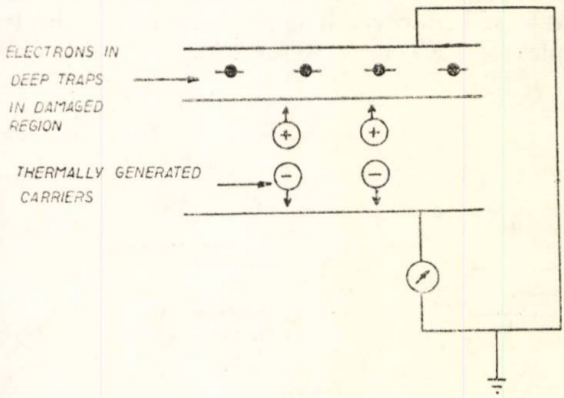

Fig. 9

3. Conclusion. The present work confirms the presence of three of the trapping levels previously reported in PET. The density of traps has been estimated to be at least 10^{17} cm^{-3}. Further work is necessary to clarify the nature and location of these traps.

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Technical Faculty of Prishtine, Yugoslavia

REFERENCES

1. Sessler G. M., West J. E., Appl. Phys. Lett., 17, 507 (1970).
2. Van Turnhout J., Polymer J., 2, 173 (1971).
3. Perlman M. M., Creswell R. A., J. Phys., 42, 531 (1971).
4. Perlman M. M., J. Electrochem. Soc., 119, 892 (1972).
5. Perlman M. M., Unger S., J. Appl. Phys., 5, 2115 (1972).
6. Creswell R. A., Perlman M. M., Kabayama M. A., Dielectric properties of Polymers, *295* (1972).
7. Abazi A. Y., *Elektronsko Bombardiranje Polimera.* Kongres Matematičara, Fizičara i Astronauta Jugoslavije, Novi Sad, 1975.

SPECTRUL CURENTULUI DE TERMOSTIMULARE LA PELICULE DE POLIMERI ÎNCĂRCATE PRIN INJECŢIE ELECTRONICĂ

(Rezumat)

Se studiază spectrul curentului de termostimulare pentru pelicule de polimer de 23 μm încărcate prin injecţie electronică cu 10^{-7} A la 5 kV, timp de cca 3 minute. Se confirmă prezenţa a trei vîrfuri şi se discută procesele fizice care au loc.

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D. K. 538.2

DIE BEWEGUNG DER EISENFLUSSIGKEITEN IN EINEM UMLAUFENDEN MAGNETFELD

VON

C. COTAE, GH. CĂLUGĂRU, V. BĂDESCU and
EMIL LUCA

1. Einführung. Die Bewegung der Eisenflüssigkeiten (feinverteilte Aufschwemmungen die von ferromagnetischen Partikeln in Grundflüssigkeiten stabiliziert sind) in einem umlaufenden Magnetfeld wurde sowohl in theoretischen Werke [1]...[6] als auch in Werke mit experimentellen Charakter studiert [1], [7]...[9].

So zum Beispiel behandelt das Werk von I. A. Kagan und seine Mitarbeiter [7] die Änderung der Drehrichtung der Eisenflüssigkeit bei einem gewissen Wert des umlaufenden Magnetfeld mit einer ungleichmässigen raumartigen Verteilung und in dem Werk [9] wird angegeben dass der Wert des Magnetfeldes, der die Drehrichtung der Eisenflüssigkeit umstellt, sowohl von der Magnetisierung der Eisenflüssigkeit (d.h. von der Volumenkonzentration der festen Phase) als auch von der Viskozität der Grundflüssigkeit abhängig ist.

Die Meinungen über die Ursache der Drehung einer Eisenflüssigkeit in einem umlaufenden Magnetfeld sind widersprüchlich.

In diesem Werk stellt man ein Phänomen vor, das bis jetzt noch nicht angezeigt war. Dieses Phänomen besteht aus einer doppelten Änderung der Drehrichtung einer Eisenflüssigkeit mit ferromagnetischen Partikeln, die ungleichmässige Dimenssionen haben und man bestimmt den Einfluss der Konzentration der grossen Partikelnaus der Eisenflüssigkeit auf die Rotationsfrequenz der Eisenflüssigkeit und auf die kritischen Felder, bei denen die Änderung der Drehrichtung stattfindet.

2. Experimentelle Technik. Die Eisenflüssigkeitproben, die in den experimentellen Messungen benutzt wurden, enthalten Partikeln von verschiedenen Grössen in verschiedenen Konzentrationen und diese Proben bestehen aus Aufschwemmnungen von Fe_2O_4 in Erdöl ($\eta = 2{,}25 \cdot 10^{-3}$ Ns/m^2) und Transformatorenöl ($\eta = 22 \cdot 10^{-3}$ Ns/m^2). Die Volumenkonzentration der magnetischen Partikeln aus diesen Proben ist $\varepsilon_s = 0{,}01$ und die Magnetisierungsstärke ist $M = 4 \,.\, 10^3$ A/m. Dafür wurden vier Arten von Eisenflüssigkeiten vorbereitet. Bei diesen Eisenflüssigkeiten haben die magnetischen Partikeln verschiedene Mitteldimensionen ($d_A = 200$ Å ; $d_B = 1\,000$ Å ; $d_C = 800$ Å ; $d_D = 600$ Å).

Durch eine paarweise Zusammenstellung dieser Eisenflüssigkeiten (A, B, C, D) in verschiedenen Proportionen haben wir Eisenflüssigkeitproben erhalten, die magnetische Partikeln von zwei verschiedenen Grössen in Proportion von 0 ; 20 ; 50 ; 80 und 100% enthalten.

So wurden die Proben aus der Tabelle 1 erhalten. Die Proben $I - V$ wurden aus den Proben $(A_0 - C_{100}) - (A_{100} - C_0)$ durch die Flockung der magnetischen Partikeln aus Erdöl und durch den Wechsel des Erdölgrundes durch Transformatorenöl erhalten.

Tabelle 1

Probe	A%	B%	Probe	A%	C%
$A_0 - B_{100}$	0	100	$A_0 - C_{100}$	0	100
$A_{20} - B_{80}$	20	80	$A_{20} - C_{80}$	20	80
$A_{50} - B_{50}$	50	50	$A_{50} - C_{50}$	50	50
$A_{80} - B_{20}$	80	20	$A_{80} - C_{20}$	80	20
$A_{100} - B_0$	100	0	$A_{100} - C_0$	100	0
Probe	**A%**	**D%**	**Probe**	**A%**	**C%**
$A_0 - D_{100}$	0	100	I	0	100
$A_{20} - D_{80}$	20	80	II	20	80
$A_{50} - D_{50}$	50	50	III	50	50
$A_{80} - D_{20}$	80	20	IV	80	20
$A_{100} - D_0$	100	0	V	100	0

Die Eisenflüssigkeitproben wurden in einem zylindrischen Glasgefäss untergebracht. Der Durchmesser des Gefässes beträgt 60 mm und die Höhe 20 mm. Dieses Gefäss befindet sich linienflüchtig im Innere eines Stators eines asynchronischen Motors mit dreiphasigen Wicklung. So entsteht ein umlaufendes Magnetfeld, der auch axial auch radial eine ungleichmässige räumliche Verteilung hat. Die freie Fläche der Eisenflüssigkeit befindet sich in der Zone von axial Höchstfeld. Die Erforschungen der Drehung der freien Fläche fand mit Hilfe von drei dielektrischen Ringflächen, deren Halbdurchmesser 8, 11 und 15 mm beträgt, in verschiedenen Gebieten von Radialfeld statt.

Weil im Inneren der Probe die Verteilung des umlaufenden Magnetfeldes ungleichmässig ist, alle Schaubilder stellen die Abhängigkeit der Rotationsfrequenz von dem Strom I aus dem Stator vor. Es wird nicht die Abhängigkeit der Rotationsfrequenz von der Stärke des Magnetfeldes H vorgestellt. (Die Abhängigkeit zwischen dem Strom aus dem Stator und dem Magnetfeld ist linear weil der Stator nicht die magnetische Sättigung erreicht.)

3. Experimentelle Ergebnisse und Diskussionen. In der Abb. 1 ist die Abhängigkeit der Rotationsfrequenz einer Ringfläche mit einem Halbdurchmesser von 15 mm von dem Wert des umlaufenden Magnetfeldes für die Proben $A-B$ dargestellt. Es wurde festgestellt, dass bei einer Radialentfernung von 15 mm gegenüber von Zentrum, die Drehrichtung der Eisenflüssigkeit zwei Richtungsänderungen hat, nicht nur eine Änderung,

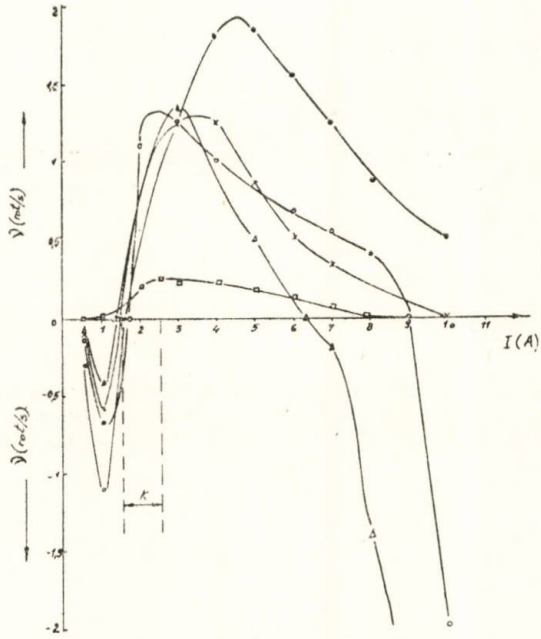

Abb. 1. — Die Abhängigkeit der Rotationsfrequenz von dem umlaufenden Magnetfeld I für die Proben: • $(A_0 - B_{100})$; × $(A_{20} - B_{80})$; ∘ $A_{50} - B_{50})$; $\Delta(A_{80} - B_{20})$ und □ $(A_{100} - B_0)$.

wie in dem Werk [7] angezeigt wurde. Das Feld, bei dem die erste Änderung der Drehrichtung stattfindet, hat ungefähr den selben Wert für alle $A-B$ Proben. Das Feld bei dem die zweite Richtungsänderung stattfindet, hat verschiedene Werte, die gleichzeitig mit der Erhöhung der Konzentration von grossen Partikeln ansteigt. Man stellt fest, dass für positive Geschwindigkeit (d. h. die selbe Richtung mit der Richtung des umlaufenden Magnetfeldes) die Höchstgeschwindigkeit von Proben, die die grösste Konzentration von grossen Partikeln A_0-B_{100} haben, erreicht wird. Nach der zweiten Richtungsänderung alle Eisenflüssigkeitproben haben eine plötzliche Steigerung der Rotationsfrequenz. I. A. Kagan und seine Mitarbeiter [7] haben die Drehung nur in dem Gebiet des Magnetfeldes erforscht (Abb. 1) und sie haben nur eine Änderung der Drehrichtung der Eisenflüssigkeit festgestellt. Zum Unterschied haben wir fest-

gestellt dass $\nu = f(H)$ zwei Höchstwerte und zwei Richtungsänderungen hat.

So wie aus der Tabelle 1 hervorgeht, die Proben $A-C$ und $A-D$ enthalten die selben Proportionen von grossen Partikeln in Aufschwemmung aber die Dimenssionen sind kleiner als in den Proben $A-B$. Die

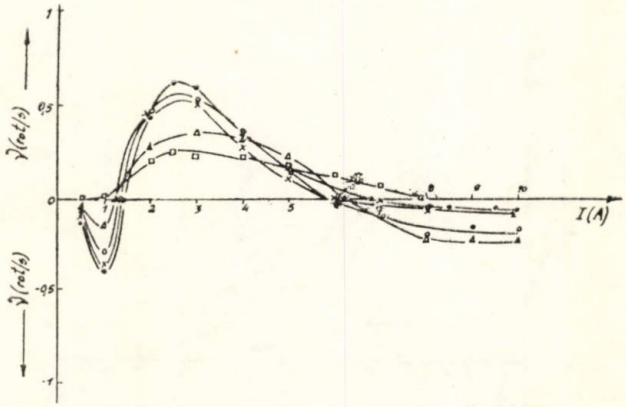

Abb. 2. — Die Abhängigkeit der Rotationsfrequenz von dem umlaufenden Magnetfeld I für die Proben: • (A_0-C_{100}), × $(A_{20}-C_{80})$; o $(A_{50}-C_{50})$; Δ $(A_{80}-C_{20})$; □ $(A_{100}-C_0)$.

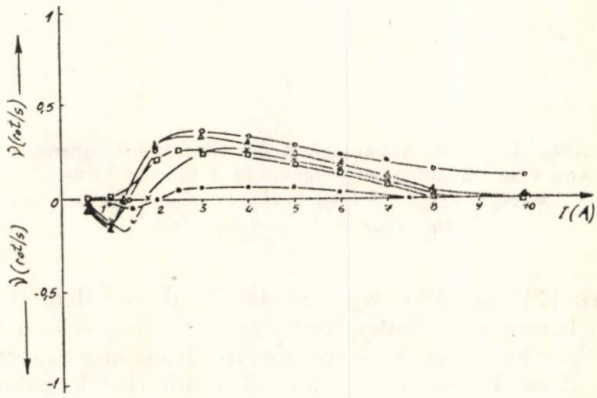

Abb. 3. — Die Abhängigkeit der Rotationsfrequenz von dem umlaufenden Magnetfeld I für die proben: • (A_0-D_{100}); × $(A_{20}-D_{80})$; o $(A_{50}-D_{50})$; Δ $(A_{80}-D_{20})$; □ $(A_{100}-D_0)$.

Abhängigkeit der Rotationsfrequenz von dem umlaufenden Magnetfeld der Proben $A-C$ und $A-D$ ist in der Abb. 2 und 3 dargestellt. Es ist zu bemerken, dass bei den $A-D$ Proben (Abb. 3) die zweite Richtungsänderung verschwindet. Daraus ergibt sich dass die Differenz zwischen den Dimenssionen der Partikeln kleiner ist, also die Eisenflüssigkeit ist

homogen, hat die Rotationsfrequenz, bei jenem Feld, kleinere Werte und die zweite Änderung der Drehrichtung findet nicht mehr statt.

Die Abb. 4 stellt die Abhängigkeit der Rotationsfrequenz von dem Magnetfeld I bei drei Werte der Strahldistanz gegenüber vom Zentrum (R = 8, 11, 15 mm) für die Eisenflüssigkeit $A_{50}-C_{50}$ dar.

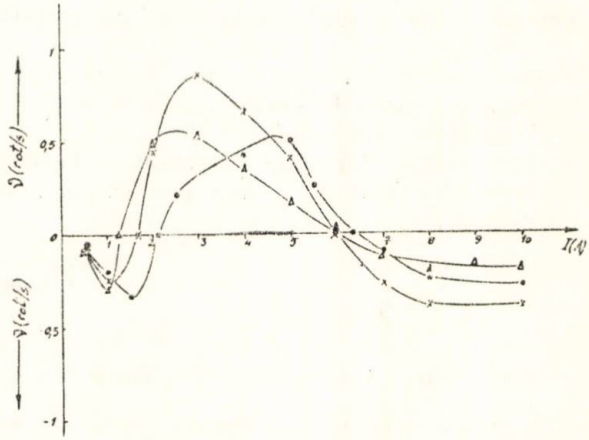

Abb. 4. — Die Abhängigkeit der Rotationsfrequenz von dem unlaufenden Magnetfeld für drei radiale Werte gegenüber von Zentrum: • R = 8 mm, × R = 11 mm, Δ R = 15 mm für die Probe (A_{50} – C_{50}).

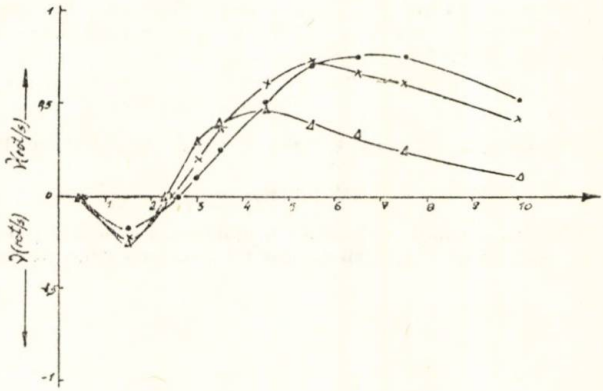

Abb. 5. — Die Abhängigkeit der Rotationsfrequenz von dem unlaufenden Magnetfeld I für drei radiale Werte gegenüber von Zentrum: • R = 8 mm, × R = 11 mm, Δ R = 15 mm für die Probe III.

Die Abb. 5 stellt die selbe Abhängigkeit der Rotationsfrequenz von dem Magnetfeld dar, wie in der Abb. 4 aber die Proben III haben die selbe Konzentration wie $A_{50}-C_{50}$ nur die Grundflüssigkeit (Transformatorenöl) hat eine grössere Viskozität. Wenn wir die Abb. 4 und 5 vergleichen er-

gibt sich hervor dass der Wert des Feldes, bei dem die Frequenz den Höchswert nach der ersten Richtungsänderung hat, grösser ist wenn die Viskozität der Grundflüssigkeit grösser ist. Ebenfalls beide Felder, bei denen die Richtungsänderungen stattfinden, sind gegen grössere Werte verrückt wenn die Viskozität der Grundflüssigkeit grösser ist.

Wir ziehen in Betracht, dass die Forschungen fortgesetzt werden müssen um das Verhalten der Eisenflüssigkeit in einem umlaufenden Magnetfeld erklären zu können.

4. Schlussfolgerungen. 1. Es wurde zum ersten mal eine doppelte Änderung der Drehrichtung einer Eisenflüssigkeit in einem ungleichmässigen umlaufenden Magnetfeld festgestellt.

2. Es wurde der Einfluss der ungleichmässigen Verteilung der magnetischen Partikeln und der Viskozität der Eisenflüssigkeit auf die Rotationsfrequenz dargestellt.

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Technische Hochschule, Jassy,
Lehrstuhl von Physik

LITERATURVERZEICHNIS

1. Moskowitz R., Rosenzweig R. E., Appl. Physics Letters, **11**, *10*, 301 (1967).
2. Заичев В. М., Слиомис Л. И., ПМТФ, **5**, *11* (1969).
3. Jenkins J. T., Le journal de physique, **32**, 931 (1971).
4. Гласов О. А,, М. Г., *2*, 14 (1975).
5. Теберс О. А., М. Г., *3*, 151 (1974).
6. Суязов В. М., ПМТФ, *4*, 40 (1970).
7. Каган И. Я., Рыков В. Г., Янтовский Е. И., М. Г., *2*, 135 (1973).
8. Mailfert R., Martinet A., Le journal de physique, **34**, 197 (1973).
9. Călugăru Gh., Cotae C., Bădescu R., Bădescu V., Luca E., Rev. Rom. Phys., **21**, *4*, 439 (1976).

MIŞCAREA FEROFLUIDELOR ÎNTR-UN CÎMP MAGNETIC ROTITOR

(Rezumat)

Se studiază influenţa mărimii particulelor magnetice, a concentraţiei acestora şi a vîscozităţii lichidului de bază asupra mişcării ferofluidului într-un cîmp magnetic rotitor.

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DIELECTRIC BEHAVIOUR OF Li-Zn AND Li-Cd FERRITES

BY

N. REZLESCU and EMIL LUCA

Keeping in view that Zn^{2+} ions have a clear preference for tetrahedral sites [1], in a previous work [2] we are studied the influence of substitution of Zn by Cd on Curie temperature, saturation magnetization and resistivity in ferrites with lithium.

The aim of the present work was to investigate the dielectric behaviour of same ferrites. The measurements were carried out in the frequency range 50 to 10^6 Hz and only on slowly cooled samples in previous work [2]. The dielectric constant ε_r' and the loss tangent tg δ were measured on disk-shaped samples by means of a capacity bridge. The surfaces of the disks were covered with silver by thermal evaporation in vacuum.

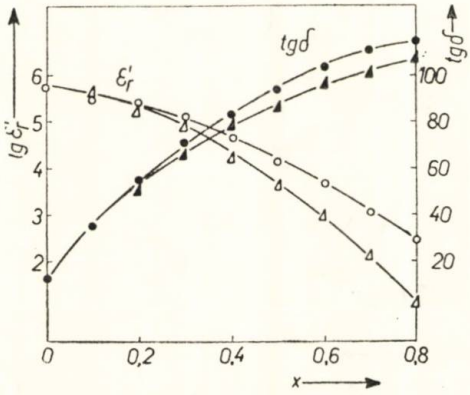

Fig. 1. — ε_r' (○△) and tg δ(● △) as a function of composition for $(Li_{0.5}Fe_{0.5})_{1-x}Me_x Fe_2O_4$ ferrites : ○ ●, Me = Zn ; △▲, Me = Cd.

Fig. 1 shows the variation of the dielectric constant ε_r' and the loss tangent tg δ as a function of composition at room temperature and at frequency of 1 kHz for Li-Zn and Li-Cd ferrites. One can observe that the dielectric constant ε_r' decreases continuously with increase of Zn or

Cd in lithium ferrite. This decrease of dielectric constant can be explained by the decrease of number of Fe^{2+} ions with the departure from Li ferrite, keeping in view the supposition that the mechanism of the polarization process in ferrites is similar to that of the conduction process [3] (By the electronic exchange $Fe^{2+} \leftrightarrows Fe^{3+}$ one obtains local displacement of electrons in the direction of the applied electrical field ; these displacements determine the polarization of the ferrite). The appearance of Fe^{2+} ions in ferrites with lithium is explained in terms of Li evaporation during sintering [4]. It is assumed that the number of cation vacancies appearing after Li loss is equal to the number of Fe^{3+} ions which become divalent (Fe^{2+}). Therefore, the diminution of Li in ferrites will determine a decrease of Fe^{2+} concentration and a decrease of ε'_r.

From Fig. 1 it may be observed also that beginning with $x = 0.2$ the values of dielectric constant ε'_r for Li-Zn ferrites are higher than those of Li-Cd ferrites. This fact is probable due to a higher concentration of Fe^{2+} ions in Li-Zn ferrites. In these ferrites we can explain the presence of a more concentration of Fe^{2+} ions in terms of Zn evaporation during sintering [1]. Contrary, for Li-Cd ferrites a similar assumption is less probable because of ionic radius of Cd^{2+} which is 20% greater than of Zn^{2+}, and the mobility and, consequently, the possibility of Cd evaporation is reduced in comparison with Zn evaporation.

Concerning the dielectric loss (Fig. 1), these increase continuously with Li diminution in mixed ferrite. One can observe the same difference as for ε'_r between the values of tg δ for Li-Zn and Li-Cd ferrites.

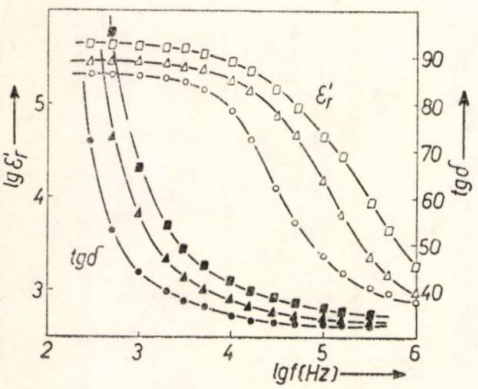

Fig. 2 — ε'_r (○ △ □) and tg δ (● △ ■) as a function of frequency at different temperatures for $Li_{0.4}Zn_{0.2}Fe_{2.3}$ ferrite ; ○ ●, $T = 22°C$; △ ▲, $T = 68°C$; □ ■, $T = 126°C$.

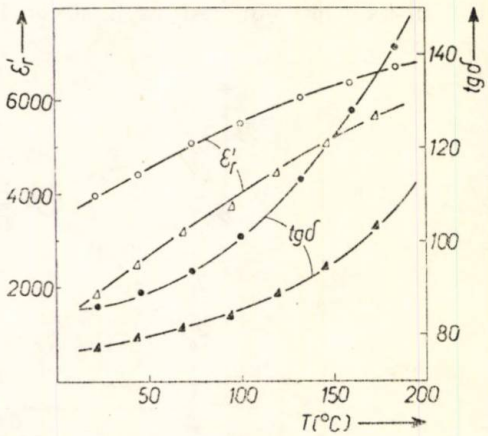

Fig. 3 — ε'_r (○ Δ) and tg δ (● ▲) as a function of temperature at different requencies for $Li_{0.25}Cd_{0.5}Fe_{2.25}O_4$ ferrite : ○ ●, $f = 1$ kHz ; △ ▲ , $f = 10$ kHz.

The dielectric behaviour as a function of frequency for Li-Zn and Li-Cd ferrites is the same, similar to the well-known frequency dependence of ε'_r for spinel ferrites [5], [6] : ε'_r decreases with increasing frequency showing a dispersion region in a certain frequency range (Fig. 2).

The dispersion region it desplaces to the higher frequencies with increase of the measuring temperature. The loss factor tg δ decreases at the beginning very rapidly with increasing frequency and then remains practically constant.

The temperature dependence of the dielectric parameters is also similar for Li-Zn and Li-Cd ferrites (Fig. 3): the ε_r' and tg δ increase continuously with increasing temperature.

In conclusion, the substitution of Zn by Cd in ferrites with lithium has no influence on the dielectric behaviour of these ferrites. Therefore, it exist the possibility of this substitution in some ferrites.

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Polytechnic Institute, Jassy,
Department of Physics

REFERENCES

1. Smit J., Wijn H. P. J., *Les Ferrites*. Bibl. Tech. Philips, Dunod, Paris, 1961.
2. Rezlescu N., Condurache D., Phys. Stat. Sol. (a), **26**, k 41 (1974).
3. Рабкин Л. И., Новикова З. И., *Ферриты*. Минск, 1960.
4. Rezlescu N., Condurache D., Petrariu P., Luca E., J. Amer. Ceram. Soc., **56** (1973).
5. Krötzsch M., Phys. Stat. Sol., **6**, 479 (1964).
6. Haberey F., Wijn H. P. J., Phys. Stat. Sol., **26**, 231 (1968).

COMPORTAREA DIELECTRICĂ A FERITELOR Li-Zn şi Li-Cd

(Rezumat)

Se măsoară constanta dielectrică şi tangenta unghiului de pierderi, urmărind influenţa înlocuirii Zn din feritele Li-Zn prin Cd asupra proprietăţilor dielectrice ale feritelor cu litiu. Se constată că această înlocuire nu modifică comportarea lor dielectrică. Zn şi Cd au, din acest punct de vedere, o influenţă similară asupra feritelor cu litiu.

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C. Z. 517. 43

O CLASĂ DE OPERATORI ŞI ECUAŢII CORESPUNZĂTOARE

II. OPERATORI DE DERIVARE DE ORDIN SUPERIOR

DE

AL. CLIMESCU şi M. GONTINEAC

Într-o lucrare anterioară [2] s-a definit o clasă de operatori liniari, operînd asupra şirurilor numerice a_n cu $n \in Z$, pe care i-am numit convenţional operatori de derivare de ordinul I. Definim operatori de ordin superior k, $k \in N$, pe mulţimea şirurilor $\{a_n\}$ cu $n \in Z$ cu valori în aceeaşi mulţime.

Definiţia 1. Numim derivată de ordin $k \in N$ a lui a_n, oare-, care $n \in Z$, în modul $-p, q$ operatorul notat $\Delta^k_{-p,q}\, a_n$ definit prin $\Delta^k_{-p,q}\, a_n = \sum_{i=-p}^{q} \alpha_i\, a_{n+i}$ unde coeficienţii $\alpha_{-p}, \alpha_{-p+1}, \ldots, \alpha_{-1}, \alpha_0, \alpha_1, \ldots, \alpha_{q-1}$ $\alpha_q \in R$ fixaţi şi satisfăcînd condiţiile :

$$(1.1)\quad \sum_{i=-p}^{q} i^j\, \alpha_i = 0 \quad \text{pentru} \quad j = 0, 1, 2, \ldots, k-1, \qquad \text{şi} \qquad (1.2)\quad \sum_{i=-p}^{q} i^k\, \alpha_i = k!$$

Şirul $\Delta^k_{-p,q}\, a_n$ il putem interpreta ca imaginea şirului iniţial printr-un operator evident liniar.

Dacă $p = 0$ $\Delta^k_q\, a_n$ se vor numi derivate de ordin k la dreapta lui a_n, iar dacă $q = 0$ $\Delta^k_{-p}\, a_n$ se vor numi derivate de ordinul k la stînga lui a_n.

Operatorii de ordin k ai lui a_n, oarecare $n \in Z$, daţi de

$$(1)\qquad d^k_m\, a_n = \frac{\sum_{i=0}^{k-1} (-1)^i\, C^i_{k-1}\, [a_{n+m+k-i-1} - a_{n+k-i-1}]}{m},$$

cu $m \in Z - \{0\}$, îi vom numi operatorii bază de ordin k ai lui a_n, denumire care va fi justificată ulterior.

Consideraţiile ulterioare se vor face aproape exclusiv pentu $n = 0$ transpunîndu-se fără dificultate pentru oarecare $n \in Z$ şi se va utiliza limbajul comod de operatori aplicaţi lui a_0.

Lema 1. *O combinaţie liniară de doi operatori de ordin k de derivare a lui a_0, avînd suma coeficienţilor 1 este de asemenea un operator de ordinul k de derivare a lui a_0.*

Fie $\Delta^k_{-p_j, q_j} a_0$ cu $j = 1, 2$ doi operatori de derivare de ordin k a lui a_0. Să arătăm că $\lambda \Delta^k_{-p_1, q_1} a_0 + (1 - \lambda) \Delta^k_{-p_2, q_2} a_0$ este tot un operator de ordin k a lui a_0. Conform Definiţiei 1 $\Delta^k_{-p_j q_j} a_0 = \sum_{i=-p_j}^{q_j} \alpha_{ij} a_i$, coeficienţii α_{ij} satisfăcînd condiţiile

$$(2.1) \quad \sum_{i=-p_j}^{q_j} i^m \alpha_{ij} = 0 \quad \text{pentru} \quad m = 0, 1, \ldots, k-1 \quad \text{şi} \quad (2.2) \quad \sum_{i=-p_j}^{q_j} i^k \alpha_{ij} = k!$$

unde $j = 1, 2$. Atunci

$$\lambda \Delta^k_{-p_1, q_1} a_0 + (1 - \lambda) \Delta^k_{-p_2, q_2} a_0 = \lambda \sum_{i=-p_1}^{q_1} \alpha_{i1} a_i + (1 - \lambda) \sum_{i=-p_2}^{q_2} \alpha_{i2} a_i.$$

Să arătăm că coeficienţii lui a_i satisfac condiţiile de tip (1.1), (1.2),

$$\lambda \sum_{i=-p_1}^{q_1} i^m \alpha_{i1} + (1 - \lambda) \sum_{i=-p_2}^{q_2} i^m \alpha_{i2} = 0$$

cu $m = 0, 1, \ldots, k-1$ datorită condiţiilor (2.1).

$$\lambda \sum_{i=-p_1}^{q_1} i^k \alpha_{i1} + (1 - \lambda) \sum_{i=-p_2}^{q_2} i^k \alpha_{i2} = \lambda k! + (1 - \lambda) k! = k!$$

datorită condiţiilor (2.2).

P r o p o z i ţ i a 1. *O combinaţie liniară de s, cu $s \in N$, operatori de ordin k a lui a_0, avînd suma coeficienţilor 1 este un operator de ordinul k de derivare a lui a_0.*

Fie $\Delta^k_{-p_j, q_j} a_0$ cu $j = 1, 2, \ldots, s$, s operatori de ordinul k a lui a_0. Să arătăm că $\sum_{j=1}^{s} \lambda_j \Delta^k_{-p_j, q_j} a_0$ cu $\sum_{j=1}^{s} \lambda_j = 1$ este un operator de ordinul k a lui a_0.

Vom demonstra prin inducţie. Pentru $s = 2$ conform Lemei 1 propoziţia este adevărată. Presupunem că $\sum_{j=1}^{s-1} \lambda'_j \Delta^k_{-p_j, q_j} a_0$ cu $\sum_{j=1}^{s-1} \lambda'_j = 1$ este un operator de ordin k de derivare a lui a_0. Atunci

$$\sum_{j=1}^{s} \lambda_j \Delta^k_{-p_j, q_j} a_0 = \sum_{j=1}^{s-1} \lambda_j \Delta^k_{-p_j, q_j} a_0 + \lambda_s \Delta^k_{-p_s, q_s} a_0 = \sum_{j=1}^{s-1} \mu \lambda'_j \Delta^k_{-p_j, q_j} a_0 +$$

$$+ \lambda_s \Delta^k_{-p_s, q_s} a_0 = \mu \sum_{j=1}^{s-1} \lambda'_j \Delta^k_{-p_j, q_j} a_0 + \lambda_s \Delta^k_{-p_s, q_s} a_0$$

unde

$$(3) \qquad \lambda'_j = \frac{\lambda_j}{\mu} \quad \text{şi} \quad \mu = \sum_{j=1}^{s-1} \lambda_j \quad \text{cu ipoteza} \quad \sum_{j=1}^{s-1} \lambda_j \neq 0$$

care nu restrînge generalitatea. Din (3) rezultă $\sum_{j=1}^{s-1} \lambda'_j = 1$ şi deci conform ipotezei inductive $\sum_{j=1}^{s-1} \lambda'_j \Delta^k_{-p_j, q_j} a_0$ este un operator de ordinul k a lui a_0. Evident

$\mu + \lambda_s = 1$ şi atunci conform Lemei 1 $\sum_{j=1}^{s} \lambda_j \Delta^k_{-p_j, q_j} a_0$ este un operator de ordin k a lui a_0.

P r o p o z i ţ i a 2. *Operatorii de ordin 2 a lui a_0 se obţin prin aplicarea a doi operatori de ordinul 1 lui a_0.*

Considerăm şirul a_n cu oarecare $n \in Z$. Fiecărui termen al şirului îi aplicăm operatorul $\Delta_{-p,q}$ de ordin unu, $\Delta_{-p,q}\, a_n = \sum_{i=-p}^{q} \alpha_i\, a_{n+i}$ oarecare $n \in Z$, coeficienţii α_i satisfăcînd condiţiile (1.1) şi (1.2) pentru $k = 1$. Obţinem şirul ..., $\Delta_{-p,q}\, a_{-p}$, $\Delta_{-p,q}\, a_{-p+1}$,..., $\Delta_{-p,q}\, a_{-1}$, $\Delta_{-p,q}\, a_0$, $\Delta_{-p,q}\, a_1$,..., ..., $\Delta_{-p,q}\, a_q$,... Aplicăm termenului $\Delta_{-p,q}\, a_0$ operatorul $\Delta_{-r,s}$ obţinînd:

$$\Delta_{-r,s}\,(\Delta_{-p,q}\, a_0) = \sum_{j=-r}^{s} \beta_j\, \Delta_{-p,q}\, a_j$$

unde

$$(3.1) \qquad \sum_{j=-r}^{s} \beta_j = 0 \quad \text{şi} \qquad\qquad (3.2) \qquad \sum_{j=-r}^{s} j\, \beta_j = 1.$$

Deci $\Delta_{-r,s}\,(\Delta_{-p,q}\, a_0) = \sum_{j=-r}^{s} \beta_j \sum_{i=-p}^{q} \alpha_i\, a_{j+i} = \sum_{j=-r}^{s} \sum_{i=-p}^{q} \beta_j\, \alpha_i\, a_{j+i}$.

Pentru ca $\Delta_{-r,s}\,(\Delta_{-p,q}\, a_0)$ să fie un operator de ordin 2 va trebui să arătăm că coeficienţii α_i, β_j satisfac condiţiile (1.1), (1.2) pentru $k = 2$.

$$\sum_{j=-r}^{s} \sum_{i=-p}^{q} \beta_j\, \alpha_i = \sum_{j=-r}^{s} \beta_j \sum_{i=-p}^{q} \alpha_i = 0 \quad \text{din (1.1) pentru} \quad k = 1.$$

$$\sum_{j=-r}^{s} \sum_{i=-p}^{q} (i+j)\, \beta_j \alpha_i = \sum_{j=-r}^{s} \beta_j \sum_{i=-p}^{q} i\, \alpha_i + \sum_{j=-r}^{s} j\, \beta_j \sum_{i=-p}^{q} \alpha_i = \sum_{j=-r}^{s} \beta_j + \sum_{i=-p}^{q} \alpha_i = 0$$

din (1.1), (1.2) pentru $k = 1$ şi (3.1), (3.2).

$$\sum_{j=-r}^{s} \sum_{i=-p}^{q} (i+j)^2\, \beta_j\, \alpha_i = \sum_{j=-r}^{s} \sum_{i=-p}^{q} i^2\, \beta_j\, \alpha_i + 2 \sum_{j=-r}^{s} \sum_{i=-p}^{q} i\, j\, \beta_j\, \alpha_i + \sum_{j=-r}^{s} \sum_{i=-p}^{q} j^2\, \beta_j\, \alpha_i =$$

$$= \sum_{j=-r}^{s} \beta_j \sum_{i=-p}^{q} i^2\, \alpha_i + 2 \sum_{j=-r}^{s} j\, \beta_j \sum_{i=-p}^{q} i\, \alpha_i + \sum_{j=-r}^{s} j^2\, \beta_j \sum_{i=-p}^{q} \alpha_i = 2.$$

Am folosit (1.1), (1.2) pentru $k = 1$ şi (3.1), (3.2).

Observaţia 1. Deci $\Delta_{-r,s}\,(\Delta_{-p,q}\, a_0) = \Delta^2_{-r-p,\, s+q}\, a_0$ şi deoarece expresia constructivă a lui $\Delta_{-r-p,s+q}\, a_0$ este o sumă dublă, care poate fi calculată în două moduri, ajungem la concluzia că aplicarea operatorilor este permutabilă.

Observaţia 2. Propoziţia se poate extinde pentru aplicare a k operatori de ordinul 1 a lui a_0.

Operatorului $\Delta^k_{-p,q}\, a_0$ îi facem să corespundă funcţia raţională $F^k_0(x) = \sum_{i=-p}^{q} \alpha_i\, x^i$ cu $x \in R - \{0\}$ care satisface evident condiţiile (1.1) şi (1.2).

Orice funcţie raţională $F^k_0(x) = \sum_{i=-p}^{q} \alpha_i\, x^i$ cu $x \in R - \{0\}$ satisfăcînd condiţiile

(4.1) $F_0^k(1) = 0,$ (4.2) $\dfrac{d^i F_0^k}{dx^i}(1) = 0,$

cu $i = 1, 2, \ldots k - 1$ şi

(4.3) $$\frac{d^k F_0^k}{dx^k}(1) = k!$$

corespunde operatorului $\Delta^k_{-p,q}\, a_0$. Funcţiile raţionale

(5) $$f_m^k(x) = \frac{(x-1)^{k-1}(x^m - 1)}{m},$$

cu $x \in R - \{0\}$, $m \in Z - \{0\}$ şi $k \in N$ corespund operatorilor bază $d_m^k\, a_0$. Operatorului $\Delta^k_{-p,q}\, a_n$ îi corespunde funcţia raţională $F_n^k(x) = x^n F_0^k(x)$.

P r o p o z i ţ i a 3. *Produsul a două funcţii raţionale corespunzătoare la doi operatori de ordinul unu de derivare a lui a_0 este o funcţie raţională corespunzătoare unui operator de derivare de ordin 2 a lui a_0, operator care se obţine ca în Propoziţia 2.*

Fie $F_0(x) = \sum_{i=-p}^{q} \alpha_i x^i$ satisfăcînd condiţiile $F_0(1) = 0$ şi $\dfrac{dF_0}{dx}(1) = 1$ şi $G_0(x) = \sum_{j=-r}^{s} \beta_j x^j$ satisfăcînd condiţiile $G_0(1) = 0$ şi $\dfrac{dG_0}{dx}(1) = 1$ două funcţii raţionale corespunzătoare operatorilor de ordin unu $\Delta_{-p,q}\, a_0$ şi $\Delta_{-r,s}\, a_0$.

Atunci $F_0(x)\, G_0(x) = \sum_{i=-p}^{q} \sum_{j=-r}^{s} \alpha_i \beta_j x^{i+j} = F_0^2(x)$. Să arătăm că este o funcţie raţională corespunzătoare unui operator de ordin doi a lui a_0.

$F_0^2(1) = \sum_{i=-p}^{q} \sum_{j=-r}^{s} \alpha_i \beta_j = \sum_{i=-p}^{q} \alpha_i \sum_{j=-r}^{s} \beta_j = 0$ datorită lui (4.1) pentru $k = 1$

$\dfrac{dF_0^2}{dx}(1) = \sum_{i=-p}^{q} \sum_{j=-r}^{s} (i+j)\, \alpha_i \beta_j = 0$ datorită lui (4.2) pentru $k = 1$;

$$\frac{d^2 F_0^2}{dx^2}(1) = \sum_{i=-p}^{q} \sum_{j=-r}^{s} (i+j)(i+j-1)\, \alpha_j \beta_j = \sum_{i=-p}^{q} i^2 \alpha_i \sum_{j=-r}^{s} \beta_j +$$

$$+ 2 \sum_{i=-p}^{q} i\, \alpha_i \sum_{j=-r}^{s} j\, \beta_j + \sum_{i=-p}^{q} \alpha_i \sum_{j=-r}^{s} j^2 \beta_j - \sum_{i=-p}^{q} \sum_{j=-r}^{s} (i+j)\, \alpha_i \beta_j = 2,$$

datorită condiţiilor (4.1), (4.2), (4.3) pentru $k = 1$.

P r o p o z i ţ i a 4. *Orice funcţie raţională $F_0^k(x) = \sum_{i=-p}^{q} \alpha_i x^i$ cu $x \in R - \{0\}$ şi satisfăcînd condiţiile* (4.1), (4.2), (4.3) *se poate scrie ca un produs de k funcţii raţionale corespunzătoare unui operator de ordin unu a lui a_0.*

Datorită condiţiilor (4.1), (4.2), (4.3) $F_0^k(x) = (x-1)^k\, \Phi(x)$ cu $\Phi(1) \neq 0$,

$$\frac{d^k F_0^k(x)}{dx^k} = k!\, \Phi(x) + C_k^1\, k(k-1) \ldots 2\, (x-1) \frac{d\Phi(x)}{dx} + \ldots$$

$$\ldots + C_k^{k-1}\, k(k-1)(x-1)^{k-1} \frac{d^{k-1}\Phi(x)}{dx^{k-1}} + (x-1)^k \frac{d^k \Phi(x)}{dx^k}.$$

Atunci pentru $x = 1$ obţinem $\frac{d^k F_0^k}{dx^k}(1) = k!\,\Phi(1)$, dar din (4.3) rezultă $k!\,\Phi(1) = k!$ şi deci $\Phi(1) = 1$.

$F_0^k(x) = (x-1)^{k-1}[(x-1)\Phi(x)]$; notăm $f_1(x) = x - 1$ şi

$$F_0(x) = (x-1)\Phi(x), \quad \frac{dF_0(x)}{dx} = \Phi(x) + (x-1)\frac{d\Phi(x)}{dx},$$

$$\text{atunci } F_0(1) = 0 \text{ şi } \frac{dF_0}{dx}(1) = \Phi(1) = 1.$$

Din condiţiile satisfăcute de $F_0(x)$ şi deoarece $f_1(x) = x - 1$ rezultă că $f_1(x)$ şi $F_0(x)$ sînt funcţii raţionale corespunzătoare unui operator de ordin unu a lui a_0.

Propoziţia 5. *Produsul a k funcţii raţionale* $\varphi_1(x), \varphi_2(x), \ldots, \varphi_k(x)$ *cu* $x \in R - \{0\}$ *corespunzătoare la k operatori de ordin unu, este o funcţie raţională corespunzătoare unui operator de ordin k.*

Din ipoteză rezultă

$$(6.1) \qquad \varphi_i(1) = 0 \qquad \text{şi} \qquad (6.2) \qquad \frac{d\varphi_i}{dx}(1) = 1, \quad i = 1, 2, \ldots, k.$$

Să arătăm că $F_0^k(x) = \varphi_1(x)\,\varphi_2(x)\ldots\varphi_k(x)$ satisface condiţiile (4.1), (4.2), (4.3). Vom demonstra prin inducţie. Pentru $k = 2$, $F_0^2(x) = \varphi_1(x)\,\varphi_2(x)$. Atunci

$$\frac{d\,F_0^2(x)}{dx} = \varphi_2(x)\frac{d\varphi_1(x)}{dx} + \varphi_1(x)\frac{d\varphi_2(x)}{dx}$$

şi

$$\frac{d^2 F_0^2(x)}{dx^2} = \varphi_2(x)\frac{d^2\varphi_1(x)}{dx^2} + 2\frac{d\varphi_1(x)}{dx}\frac{d\varphi_2(x)}{dx} + \varphi_1(x)\frac{d^2\varphi_2(x)}{dx^2}.$$

Din (6.1) şi (6.2) rezultă $F_0^2(1) = 0$, $\frac{dF_0^2}{dx}(1) = 0$, $\frac{d^2F_0^2}{dx}(1) = 2$ şi deci $F_0^2(x)$ este o funcţie raţională corespunzătoare unui operator de ordin doi. Presupunem că $F_0^{k-1}(x) = \varphi_1(x)\,\varphi_2(x)\ldots\varphi_{k-1}(x)$ este o funcţie raţională corespunzătoare unui operator de ordin $k - 1$, deci satisface condiţiile

$$(7.1) \qquad F_0^{k-1}(1) = 0, \qquad (7.2) \qquad \frac{d^i F_0^{k-1}}{dx^i}(1) = 0, \quad i = 1, 2, \ldots, k-1$$

$$(7.3) \qquad \frac{d^{k-1}F_0^{k-1}}{dx^{k-1}}(1) = (k-1)!$$

Atunci $F_0^k(x) = F_0^{k-1}(x)\varphi_k(x)$ şi $F_0^k(1) = 0$ deci este satisfăcută condiţia (4.1).

$$\frac{d^i F_0^k(x)}{dx^i} = \frac{d^i}{dx^i}[F_0^{k-1}(x)\,\varphi_k(x)] = \sum_{j=0}^{i} C_i^j \frac{d^{i-j}F_0^k(x)}{dx^{i-j}}\frac{d^j\varphi_k(x)}{dx^j} \quad \text{cu } i = 1, 2, \ldots k.$$

Pentru $i = 1, \ldots, k-1$ şi $x = 1$, avem : $\frac{d^i F_0^k}{dx^i}(1) = \sum_{j=0}^{i} C_i^j \frac{d^{i-j}F_0^k}{dx^{i-j}}(1)\frac{d^j\varphi_k}{dx^j}(1)$

şi datorită condiţiilor (6.1), (6.2), (7.1), (7.2), obţinem $\frac{d^i F_0^k}{dx^i}(1)=0,\ i=1,\dots,k-1$ deci satisfăcute condiţiile (4.2). Pentru $i=k$ şi $x=1$ avem : $\frac{d^k F_0^k}{dx^k}(1) = \sum_{j=0}^{k} C_k^j \frac{d^{k-j}F_0^k}{dx^{k-j}}(1)\frac{d^j\varphi_k}{dx^j}(1)$ şi datorită condiţiilor (6.1), (6.2), (7.1), (7.2), (7.3), obţinem $\frac{d^k F_0^k}{dx^k}(1) = C_k^1 (k-1)! = k!$ deci satisfăcută condiţia (4.3).

P r o p o z i ţ i a 6. *Orice funcţie raţională $F_k^0(x)$ satisfăcînd condiţiile* (4.1), (4.2), (4.3) *se poate exprima ca o combinaţie liniară unică cu suma coeficienţilor 1 de funcţii raţionale bază $f_m^k(x)$ date de* (5).

Din Propoziţia 2 rezultă că $F_0^k(x) = (x-1)^{k-1}F_0(x)$ unde $F_0(x)$ este o funcţie raţională corespunzătoare unui operator de ordin unu şi deci $F_0(x) = \sum_{i=-p}^{q} \beta_i x^i$ satisfăcînd condiţiile $F_0(1) = 0$ şi $\frac{dF_0}{dx}(1)=1$. Atunci coeficienţii $\beta_i \in R$ satisfac condiţiile

$$(8.1)\qquad \sum_{i=-p}^{q} \beta_i = 0 \qquad \text{şi} \qquad (8.2)\qquad \sum_{i=-p}^{q} i\,\beta_i = 1.$$

Din (5) $f_m^k(x) = \frac{(x-1)^{k-1}(x^m-1)}{m}$ de unde $x^m = \frac{m f_m^k(x)}{(x-1)^{k-1}} + 1$. Atunci

$$F_0^k(x) = (x-1)^{k-1}\sum_{i=-p}^{q}\beta_i x^i = (x-1)^{k-1}\sum_{i=-p}^{q}\beta_i\left[\frac{i f_i^k(x)}{(x-1)^{k-1}}+1\right] = \sum_{i=-p}^{q} i\,\beta_i f_i^k(x) +$$

$$+ (x-1)^{k-1}\sum_{i=-p}^{q}\beta_i = \sum_{i=-p}^{q} i\,\beta_i f_i^k(x).$$ Am ţinut seama de (8.1).

Din (8.2) rezultă că $\sum_{i=-p}^{q} i\beta_i = 1$ şi deci $F_0^k(x) = \sum_{i=-p}^{q}\gamma_i f_i^k(x)$ cu $\sum_{i=-p}^{q}\gamma_i = 1$.

Unicitatea descompunerii rezultă din proprietatea funcţiilor raţionale de acest tip care sînt liniar independente.

Observaţia 1. Propoziţia 6 justifică denumirea pentru $d_m^k a_0$ de bază. Ca şi pentru operatorii de ordin unu există o infinitate de baze.

Considerăm şirurile de pas h notate a_{nh} cu $n \in Z$ şi $h > 0$ oarecare. Un operator de derivare de ordin k în modul $-p$, q a lui a_{0h} va fi $\Delta_{-p,q}^k a_{0h} = \sum_{i=-p}^{q} \alpha_i a_{ih}$, coeficienţii $\alpha_i \in R$ satisfăcînd condiţiile:

$$(9.1)\qquad \sum_{i=-p}^{q} i^j \alpha_i = 0 \text{ cu } j=0,1,\dots,k-1 \qquad \text{şi} \qquad (9.2)\qquad \sum_{i=-p}^{q} i^k \alpha_i = \frac{k!}{h^k}.$$

Derivatele de ordin k bază a lui a_{0h} sînt :

$$d_m^k a_{0h} = \frac{\sum_{i=0}^{k-1} C_{k-1}^i \left[a_{nh+k-i-1} - a_{k-i-1}\right]}{m\,h^k}.$$

Fracţiile raţionale $F^k_{0h}(x) = \sum_{i=-p}^{q} \alpha_i x^i$ cu $x \in R - \{0\}$ satisfăcînd

$$F^k_{0h}(1) = 0, \tag{10.1}$$

$$\frac{d^i F^k_{0h}}{dx^i}(1) = 0 \quad \text{cu} \quad i = 1, 2, \ldots, (k-1) \quad \text{şi} \tag{10.2}$$

$$\frac{d^k F^k_{0h}}{dx^k}(1) = \frac{k!}{h^k} \tag{10.3}$$

corespund operatorilor $\Delta^k_{-p,q}\, a_{0h}$.

Operatorilor de ordin k bază a lui a_0, $d^k_m\, a_{0h}$ le ataşăm funcţiile raţionale $f^k_{mh}(x) = \dfrac{(x-1)^{k-1}(x^m-1)}{m\, h^k}$ cu $m \in Z - \{0\}$ şi $x \in R - \{0\}$.

Propoziţiile demonstrate pentru pasul 1 se demonstrează analog pentru pasul h.

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Institutul politehnic, Iaşi,
Catedra de matematici

BIBLIOGRAFIE

1. Climescu Al., Gontineac M., Motreanu D., *O clasă de operatori şi ecuaţi corespunzătoare.* Bul. Inst. polit. Iaşi, XX (XXIV), *3 — 4*, s.I (1974).
2. Climescu Al., Gontineac M., *O clasă de operatori şi ecuaţii corespunzătoare.* (I) *Operatori de derivare de ordin unu.* Bul. Inst. polit. Iaşi, XXII (XXVI), *3 — 4*, s.I (1976).

A CLASS OF OPERATORS AND CORRESPONDING EQUATIONS

II. The Higher-order Derivatives

(Summary)

Developping the programme sketched in [1] we define the higher-order derivatives and establish some of their properties.

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К ИССЛЕДОВАНИЮ ПОЛИЛИНЕЙНЫХ УРАВНЕНИЙ С ЧАСТНЫМИ ПРОИЗВОДНЫМИ

I. ПОЛИНОМИАЛЬНЫЕ РЕШЕНИЯ НЕКОТОРЫХ КЛАССОВ ПОЛИЛИНЕЙНЫХ УРАВНЕНИЙ

Б. А. БОНДАРЕНКО, К. В. ЛЕН, Д. МАНЖЕРОН и М. Н. ОГЮЗТОРЕЛИ

Посвящается светлой памяти
МАУРО ПИКОНЕ,
Почетному члену Академии
Социалистической Республики
Румынии

1. В рамках замечательных калитативных скачков развития чистых и прикладных математических наук Мира, а в частности их обширнейшей области дифференциальных уравнений, непрестанно и многогранно используемых для моделлирования все более и более сложных явлений органической и неорганической природы, неоценимый и непрерывный шестидесятилетний вклад академика Мауро Пиконе в науку и технику [1] включает в себя опубликованный пол века тому назад труд [2], послуживший основной всех последующих исследований по *полигармоническим* [3], [4], [5], *политепловым* [6], [7] и *поливолновым* [8] системам уравнений, частично завершающихся теперь в ряде исследовании авторов и их сотрудников посвященных дифференциальным системам сложной структуры [9]–[18].

К таким системам принадлежат и *поливолновые* уравнения вида

(1) $K^{(n)}u(x, y) \equiv (M_\mu^{(r)} - L)^{(n)} u(x, y) = F(x, y)$, $K^{(j)} = K(K^{j-1})$, $K^{(0)} = 1$,

где $x = (x_1, x_2, ..., x_\mu) \in R^\mu$, $y = (y_1, y_2, ..., y_\nu) \in R^\nu$, $M_\mu^{(r)}$ – полная производная r-го порядка в смысле Пиконе [19]–[20], $\partial^{\mu r}/\partial x_1^r \partial x_2^r ... \partial x_\mu^r$, L – линейный дифференциальный оператор оперирующий лишь над переменными y, а $F(x, y)$ – известная непрерывная функция в области $D = R^\mu \times R^\nu$, исследованные в [21] с точки зрения операционного представления решений этих уравнений.

В настоящей заметке приводятся классы полиномиальных решений уравнения [1], пресставляющие собою определенный интерес как с теоретической так и с практической точек зрения.

2. На основании формул (2) и (3) работы [21] имеет место следующее утверждение :

I. – Пусть $y^b = y_1^{b_1} y_2^{b_2} ... y_\nu^{b_\nu}$ – одночлен. Если для некоторого $b = \{b_1, b_2, ..., b_\nu\}$ с целочисленными значениями b_j можно указать такое целое положительное число m, что $L^m y^b = 0$, то уравнение (1) имеет полиномиальные решения.

Более подробные исследования полиномиальных решений можно провести при конкретных операторах $M_{\mu}^{(r)}$ и L. Рассмотрим, например, уравнение

$$K^{(n)} u(x, y) \equiv \left(\frac{\partial^2}{\partial x_1 \partial x_2} \pm \Delta_\nu\right)^{(n)} u(x, y) = 0, \tag{2}$$

где $x = (x_1, x_2)$, $y = (y_1, y_2, \ldots, y_\nu)$, $\Delta_\nu = \partial^2/\partial y_1^2 + \partial^2/\partial y_2^2 + \ldots + \partial^2/\partial y_\nu^2$.

Если в формулах (2) и (3) работы [21] положить $r=1$, $s=0$, $L=\Delta_\nu$,

$$g(y) = m!\,\frac{y^b}{b!} \equiv m!\,\frac{y_1^{b_1}}{b_1!}\,\frac{y_2^{b_2}}{b_2!} \ldots \frac{y_\nu^{b_\nu}}{b_\nu!}, \quad |b| = b_1 + b_2 + \ldots + b_\nu = m - \alpha - 2p + 2$$

и ввести множитель $(\mp 1)^{i-p+1}$ под знак суммы, то получим систему полиномиальных решений уравнения (1) в виде

$$P_{1,0,\alpha;b}^{m,p,1}(x, y) = m! \sum_{i=p-1}^{\left[\frac{m-\alpha}{2}\right]} (\mp 1)^{i-p+1} \frac{x_1^i}{i!} \frac{x_2^{i+\alpha}}{(i+\alpha)!} \Delta_\nu^{i-p+1}\left(\frac{y^b}{b!}\right), \tag{3}$$

$$P_{1,0,\alpha;b}^{m,p,2}(x, y) = m! \sum_{i=p-1}^{\left[\frac{m-\alpha}{2}\right]} (\mp 1)^{i-p+1} \frac{x_1^{i+\alpha}}{(i+\alpha)!} \frac{x_2^i}{i!} \Delta_\nu^{i-p+1}\left(\frac{y^b}{b!}\right). \tag{4}$$

В равенствах (3) и (4) $\alpha = 0, 1, 2, \ldots, m - 2p + 2$, $p = 1, 2, \ldots, n$, причем $P_{1,0,0;b}^{m,p,1}(x, y) = P_{1,0,0;b}^{m,p,2}(x, y)$.

Полиномы (3) и (4) обладают многими удобными аналитическими и вычислительными свойствами, аналогичными свойствам H_q - полиномов, построенных и изученных в работах первого из авторов [22].

3. Для сокращения дальнейших записей рассмотрим уравнение (2) при $\nu=1$ и знаке $(-)$, т.е. уравнение

$$K^{(n)} u(x_1, x_2, y) \equiv \left(\frac{\partial^2}{\partial x_1 \partial x_2} - \frac{\partial^2}{\partial y^2}\right)^{(n)} u(x_1, x_2, y) = 0. \tag{5}$$

Полагая в равенствах (3) и (4) $\Delta_\nu = \partial^2/\partial y^2$, $b = m - \alpha - 2p + 2$, вычисляя

$$\frac{\partial^{2i-2p+2}}{\partial y^{2i-2p+2}}\left(\frac{y^{m-\alpha-2p+2}}{(m-\alpha-2p+2)!}\right) = \frac{y^{m-\alpha-2i}}{(m-\alpha-2i)!}$$

и учитывая, что

$$\frac{m!}{i!\,(i+\alpha)!\,(m-\alpha-2i)!} = \binom{m}{i,\ i+\alpha},$$

где символ $\binom{a}{b,\ c} = \frac{a!}{b!\ c!\ (a-b-c)!}$ — триномиальный коэффициент, т.е. коэффициент разложения $(x+y+z)^a$, будем иметь

$$P_{1,0,\alpha}^{m,p,1}(x_1, x_2; y) = \sum_{i=p-1}^{\left[\frac{m-\alpha}{2}\right]} \binom{i}{p-1} \binom{m}{i,\ i+\alpha} x_1^i\, x_1^{i+\alpha}\, y^{m-\alpha-2i}, \tag{6}$$

$$(7) \qquad P_{1,0,\alpha}^{m,p,2}(x_1,\ x_2;\ y) = \sum_{i=p-1}^{\left[\frac{m-\alpha}{2}\right]} \binom{i}{p-1} \binom{m}{i+\alpha,\ i} x_1^{i+\alpha}\, x_2^{i}\, y^{m-\alpha-2}.$$

Полиномы (6) и (7) представляют собой однородные относительно x_1, x_2, y степени m многочлен. Они обладают многими свойствами; приведем некоторые из свойств.

а. Представления (6) и (7) при $\alpha=0,\ 1,\ 2,\ \ldots,\ m-2p+2$ образуют $2m-4p+5$ линейно независимых полиномов, удовлетворяющих уравнения $K^{(p)}\, u\,(x_1,\ x_2;\ y)=0$ и неравенству $K^{(p-1)}\, u\,(x_1,\ x_2;\ y)\neq 0$.

б. Представления (6) и (7) при $\alpha=0,\ 1,\ 2,\ \ldots,\ m-2p+2$ и $p=1,\ 2,\ \ldots,\ n$ образуют базисное множество полиномиальных степени m решений уравнения (5), состоящее из $n\,(2m-2n+3)$ линейно независимых полиномов.

в. Для полиномов (6) и (7) справедливы следующие дифференциальные соотношения (в связи с симметрией приводятся только для полиномов (6)):

$$(8) \qquad \frac{\partial}{\partial x_1} P_{1,0,\alpha}^{m,p,1}(x_1,\ x_2;\ y)=m\, P_{1,0,\alpha+1}^{m-1,p,1}(x_1,\ x_2;\ y)+m\, P_{1,0,\alpha+1}^{m-1,p-1,1}(x_1,\ x_2,\ y),$$

$$(9) \qquad \frac{\partial}{\partial x_2} P_{1,0,\alpha}^{m,p,1}(x_1,\ x_2;\ y) = m\, P_{1,0,\alpha-1}^{m-1,p,1}(x_1,\ x_2;\ y), \quad \alpha\neq 0,$$

$$(10) \qquad \frac{\partial}{\partial x_2} P_{1,0,0}^{m,p,1}(x_1,\ x_2;\ y)=m\, P_{1,0,1}^{m-1,p,2}(x_1,\ x_2;\ y)+m\, P_{1,0,1}^{m-1,p-1,2}(x_1,\ x_2;\ y),$$

$$(11) \qquad \frac{\partial}{\partial y} P_{1,0,\alpha}^{m,p,1}(x_1,\ x_2;\ y) = m\, P_{1,0,\alpha}^{m-1,p,1}(x_1,\ x_2;\ y).$$

г. Неостанавливаясь на других многочисленных свойствах, аналогичных свойствам H_q - полиномов [5], отметим, что, повидимому можно доказать, что любое регулярное в некоторой замкнутой области D решение уравнения (5) при знаке + между операторами можно равномерно аппроксимировать полиномами (6) и (7).

д. На основе формул дифференцирования (8) и (10) легко получить рекуррентные соотношения, формулы неопределенного интегрирования по каждой переменной и т.п.

е. Для полиномов (6) и (7) можно ввести так называемые „матрицы показателей“ и „матрицы коэффициентов“ в форме несколько отличной от „матриц показателей“ и „матриц коэффициентов“ введенных в работах [5] для полигармонических полиномов и использовать их для вычислений на электронно-вычислительные машинах [22].

ж. Аналогично можно изучить полиномы (3) и (4), а также полиномы для других операторов $M_{\mu}^{(r)}$ и L.

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Мат. ин-т АН Узб. ССР,
Ташкент, СССР,
Конкордя Университет,
Монреаль, Канада,
Яскии политехническии институт и
Албертскии университет,
Эдмонтон, Канада

ЛИТЕРАТУРА

1. Picone Mauro, в сб. „Biografie e Bibliografie degli Accademici Lincei", Accademia Nazionale dei Lincei, Roma, pp. 491 — 505, 1976.
2. Picone M., a) *Sulle funzioni metaarmoniche.* Rend. Accad. Naz. Lincei, Cl. Sci. fis., mat. e nat., (6) **5**, 90 — 95 (1927); *Teoremi di media caratteristici per le soluzioni di talune equazioni alle derivate parziali.* Rend. Circ. Mat. Palermo, **51**, 25 — 37 (1927); b) *Sulla convergenza delle successioni di funzioni iperarmoniche.* Bull. Soc. Roumaine Math., **38**, 105 — 112 (1936).
3. Sbrana F., *Sulle funzioni metaarmoniche in n variabili.* Renc. Circ. Mat. Palermo, **53**, 428 — 436 (1929).
4. Nicolescu M., *Les fonctions polyharmoniques.* Hermann, Paris, 54 p., **1936**.
5. Вондаренко Б. А., *Полигармонические полиномы.* Изд-во „Фан" УзССР, Ташкент, 1968 и *Некоторые вопросы теории и применение H_q - полиномов и G_q - функций.* Bul. Inst. polit. Iaşi, **XVII (XXI)**, *3—4*, s. I (1971).
6. Nicolescu M., *Ecuaţia iterată a căldurii.* St. Cerc. Mat., **5**, *3 — 4*, 243 —332 (1954).
7. Бондаренко Б. А., *Решение задачи Коши-Николеску для полилинейных уравнений в частных производных.* Bul. Inst. polit. Iaşi, Sect. I (в печати).
8. Mangeron D., *Problèmes concernant les équations polyvibrantes.* C. r. Acad. Sci. Paris, **266 A**, 870 — 873; 976 — 979; 1050 — 1052; 1103 — 1106; 1121 — 1124 (1968); **204**, 94 — 96; 544 — 547; 1022 — 1024 (1937). См. также: Rend. Accad. Naz. Lincei, Cl. Sci. fis., mat. e nat., **6 (16)**, 305 — 310 (1932); Rend. Accad. Sci. fis., mat., Napoli, **4 (2)**, 29 —40 (1932); Giorn. Mat., **71**, 89 — 139 (1933).
9. Mangeron D., Krivoshein L. E., *New Methods of Numerical Calculation of Various Integro-differential Systems* (I, II, III). Rev. Roum. Sci. Techn., Série Méc. Appl., **9**, 1195 — 1221 (1964); **10**, 3 — 34 (1965); **11**, 733 — 756 (1966).
10. Krivoshein L E., Mangeron D., Oguztoreli M. N., *Études des systèmes différentiels possédant la structure complexe.* Bull. Soc. Roy. Sci. Liège, **43**, *7 — 10*, 512 — 516 (1974); Bull. Cl. Sci. Acad. Roy. Belgique, (5) **60**, 585 — 592; 387 — — 395 (1974).
11. Leung K. V., Oguztoreli M. N., *Numerical Solutions of Various Problems Concerning Polyvibrating or Polywave Equations of Mangeron.* Rend. Accad. Sci. fis., mat., Napoli, **4 (40)**, 52 — 57 (1973); Bull. Soc. Roy. Sci. Liège, **42**, *5 — 6*, 269 — 275 (1973); Bull. Acad. Roy. Sci. Belgique, (5) **59**, 441 — 447 (1973).
12. Krivoshein L. E., Mangeron D., Oguztoreli M. N., *Calcolo con calcolatrici elettroniche delle funzioni speciali spettanti ai sistemi polivibranti e polivibranti generalizzati ed applicazioni.* Rend. Accad. Naz. Linci., Cl. sci. fis., mat. e nat., (8) **48**, 160 — 169 (1970).
13. Mangeron D., Oguztoreli M. N., *Problema di Goursat per le equazioni polivibranti.* Rend. Accad. Naz. Lincei, Cl. sci. fis., mat e nat. (в печати).
14. Krivoshein L. E., Leung K. V., Mangeron D., Oguztoreli M. N., *Étude des solutions d'une classe d'équations intégrodifferentielles non linéaires aux opérateurs hyperboliqaes ayant pour limites des fonctionnelles intégrales.* Bull. Soc. Math. de Grece (в печати).
15. Bondarenko B. A., Leung K. V., Mangeron D., Oguztoreli M. N., *Studi concernenti le equazioni polilineari.* Alla memoria di Mauro Picone. Rend. Accad. Sci. fis., mat., Napoli (в печати).
16. Nielson G. M., *A Mangeron Theorem for Triangular Domains.* Rev. Roum. Math. pures et appl. (в печати).
17. Berezanski Ju. M., *Expansions in Eigenfunctions of Selfadjoint Operators.* Amer. Math. Soc. Transl. Monographs, Chap. IV, Addenda, p. **756, 787, 1968.**
18. Easwaran S., *A Study on Certain Higher Order Partial Differential Equations of Mangeron.* A Doctoral Dissertation. Department of Mathematics, The University of Alberta, 1972.
19. Mangeron D., Krivoshein L. E., *Problemi concernenti sistemi di equazioni integro differenziali con derivate totali di Picone.* Rend. Sem. Mat. Univ. Padova **33**, 226 — 266 (1963); **34**, 344 — 368 (1964); **35**, 341 — 364 (1965).

20. Oguztoreli M. N., Mangeron D., Easwaran S., *Problèmes aux limites pour certaines systèmes différentiels aux différences, ordinaires ou aux dérivées totales au sens de M. M. Picone.* Archivum Mathematicum, Brno, 3 (VIII), 131 – 136 (1972).

21. Bondarenko B. A., Leung K. V., Mangeron D. J., Oguztoreli M. N., *Operational Representation of Solutions of a Class of Polylinear Equations.* Notices Amer. Math. Soc., 1977, Febr.

22. Bui T. D., Leung K. V., *New Developments in Computer Techniques for Stiff Differential Equations.* Sixth Canadian Congress of Applied Mechanics, May 30 – June 3, 1977. The University of British Columbia, Vancouver, B.C. Abstracts and Programme, p. 19.

STUDIUL ECUAŢIILOR POLILINIARE CU DERIVATE PARŢIALE

I. Soluţii polinomiale pentru o clasă de ecuaţii poliliniare

(Rezumat)

Autorii expun în această lucrare unele rezultate referitoare la soluţii polinominale ale unor clase de ecuaţii poliliniare, deosebit de utile în rezolvarea problemelor de aprox imare uniformă a soluţiilor unor atare ecuaţii şi pentru programări corespunzătoare la computere.

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A REFLEXIVE SUBCATEGORY OF THE CATEGORY OF STOCHASTIC AUTOMATA

BY

VLADIMIR V. TOPENČAROV and KETTY G. PEEVA

The problem of reduction and minimization of automata are examined in category aspect in detail and already solved in broad outlines for deterministic finite automata (see [1], [2]). By the means of the category theory a full description of reduction and minimization of machines by states is given in [3]. These problems remain still open for stochastic automata due to technical and principle difficulties in adapting the methods worked out for deterministic automata to this rather delicate case. There is some doubt about the possibility of solving the general reduction problem.

The object of this paper is the problem of reduction of stochastic automata by input symbols (X-reduction), without any restrictions on the type of automaton. A basic result is the theorem of reflexivity of the cateory $\bar{A}_r^S$ of the strong X-reduced stochastic automata treated as a subcategory of the category $\bar{A}^S \subset A^S$. (Here A^S is the stochastic automata category with epimorphic state component in every morphism). The paper developes some of the ideas and results given in [6].

The terminology and the notations are as in [5] for the theory of categories and as in [4] for the algebraic theory of automata. In particular we shall denote by capital letters the categories (i.e. C, H); the class of objects for C will be denoted by C_0, the set of all morphisms of C — by C and the set of the identical morphisms which is isomorphic to C_0 — by C_0.

1. Categories of Stochastic Automata. Let M be the category of mappinges over the universe M_0. Given the sets X, Q, Y and P; let

$$VP = \{f; f\colon P \to [0,1] \in M,\ [0,1] \subset R,\ \sum_{p \in P} f(p) = 1\} \subset M.$$

A *stochastic automaton* (abbreviated *S-automaton*) is a quadruple $A = (X, Q, Y, h)$; $h\colon X \times Q \to V(Q \times Y)$ is a mapping of M_0 into M, which assigns to every pair $(x, q) \in X \times Q$ an morphism $h_{(x,q)}\colon Q \times$

$\times Y \to [0,1] \in M$, such that $\sum_{(\bar{q},y) \in Q \times Y^h(q,x)} (\bar{q}, y) = 1$ holds; X, Q and Y are finite or numerable sets namely inputs, (internal) states and outputs respectively.

For further examinations it will be important to construct a functorial definition of an S-automaton. It is identical with the above given definition of S-automaton according to [6].

Let EA be a category with two objects a, b and a morphism denoted by (a, b); let $\boxdot M$ be the category of commutative squears of M with a horizontal composition and F be the functor from EA to $\boxdot M$.

D e f i n i t i o n 1. F is a *stochastic realization of the automaton sketch* EA, if the following conditions $F((a, b)) = K \in \boxdot M$, $\alpha^{\boxdot}(K) = F(a) = i_{X \times Q}$, $\beta^{\boxdot}(K) = F(b) = h$ hold, where $i_{X \times Q}$ is the identical mapping of $X \times Q$ into M, and h is an automaton stochastic mapping.

The notions S-automaton and atochastic realization of an a automaton skech are equivalent.

Given the stochastic automata A and B; let their functors-realization of the automaton sketch EA be F_A and F_B respectively.

D e f i n i t i o n 2. The natural transformation $\varphi : F_A \to F_B$ is a *morphism of stochastic automata* (*AS-morphism*) from the S-automaton A to the S-automaton B. The category of stochastic automata (denoted by A^S) is a subcategory of Nat (EA, $\boxdot M$), which is defined by the natural transformations between functors-S-automata.

P r o p o s i t i o n 1. *Every AS-morphism is uniquely representable by an ordered triple of mappings* $\varphi_X : X \to X'$, $\varphi_Q : Q \to Q'$ *and* $\varphi_Y : Y \to Y'$ *making commutative the following diagram*

$$\begin{tikzcd} X \times Q \arrow[r, "h"] \arrow[d, "\varphi_X \times \varphi_Q"'] & V(Q \times Y) \arrow[d, "V(\varphi_Q \times \varphi_Y)"] \\ X' \times Q' \arrow[r, "h'"] & V(Q' \times Y') \end{tikzcd}$$

Here $V(\varphi_Q \times \varphi_Y)$ denotes a mapping, which is defined in the following way:

$$V(\varphi_X \times \varphi_Q)\, h_{(c,e)} = \begin{cases} h''_{(\varphi_X(c),\ \varphi_Q(e))} & \text{if } h_{c,e)} \in h(X \times Q), \\ f \text{ if } h_{(c,e)} \overline{\in} h(X \times Q), \end{cases}$$

where f is an arbitrary fixed element of $V(Q' \times Y')$.

Let X^* and Y^* be the free semigroups of words over X and Y respectively and let $h^* : X^* \times Q \to V(Q \times Y^*)$ be an extention of $h : X \times Q \to V(Q \times Y)$, which is defined as follows:

$$h^*(\Lambda, q) = h^*_{(\Lambda,q)}, \quad \text{where} \quad h^*_{(\Lambda,q)} \in V(Q \times Y^*) \text{ and}$$

$$h^*_{(\Lambda,q}(\bar{q}, y) = \begin{cases} 1 \text{ if } \bar{q} = q \text{ and } y = 1, \\ 0 \text{ otherwise}; \end{cases}$$

$$h^*(x, q) = h_{(x,q)} \quad \text{for each} \quad x \in X,$$

and with the recursive relation

$$h^*(ux, q) = h^*_{(ux,q)}, \quad h^*_{(ux,q)} \in V(Q \times Y^*),$$

$$h^*_{(ux,q)}(\bar{q}, wy) = \sum_{q \in Q} h^*_{(u,q)}(\bar{\bar{q}}, w)\, h_{(x,\bar{\bar{q}})}(\bar{q}, y).$$

Behaviour of the S-automaton $A = (X, Q, Y, h)$ is the mapping $h^* : X^* \times Q \to V(Q \times Y^*)$ canonically associated to the automaton A. We can without any difficulties construct the functorial representation of the behaviour.

Definition 3. F is a *behaviour of S-automaton* if $F(a) = i_{X^* \times Q}$, $F((a, b)) = K \in \boxed{\cdot} M$, $F(b) = h^* : X^* \times Q \to V(Q \times Y^*)$. *Morphism of behaviours* is the natural transformation $\varphi : F_A \to F_B$, defined by the ordered triple $\varphi = (\varphi_{X^*}, \varphi_Q, \varphi_{Y^*})$, so that the extension of φ_X and φ_Y to φ_{X^*} and φ_{Y^*} respectively make the following diagram commutative

$$\begin{array}{ccc} X^*_A \times Q_A & \xrightarrow{\;h^*_A\;} & V_A(Q_A \times Y^*_A) \\ \varphi_{X^*} \times \varphi_Q \Big\downarrow & & \Big\downarrow V(\varphi_Q \times \varphi_{Y^*}). \\ X^*_B \times Q_B & \xrightarrow{\;h^*_B\;} & V(Q_B \times Y^*_B) \end{array}$$

Here $\varphi_{X^*} : X^*_A \to X^*_B$ and $\varphi_{Y^*} : Y^*_A \to Y^*_B$ are defined as follows

$$\varphi_{X^*}(\Lambda) = \Lambda, \quad \varphi_{X^*}(xu) = \varphi_X(x) \cdot \varphi_X(u),$$

and by analogy for φ_{Y^*}.

The subclass of Nat $(EA, \boxed{\cdot} M^*)$, where $\boxed{\cdot} M^*$ is constitued by the morphisms of behaviours, generates the subcategory B^S of behaviours of stochastic automata.

In particular, let us consider those natural transformations of Nat $(EA, \boxed{\cdot} M^*)$, in which the morphisms $\varphi_Q : Q \to Q'$ are epimorphisms of M. They define the subcategory of B^S, denoted by $\overline{B}^S$.

For our further examinations we define a functor $H : A^S \to B^S$ as follows : given the S-automata $A = (X, Q, Y, h)$ and $A' = (X', Q', Y', h')$ from A^S_0 and the AS-morphism $\varphi : A \to A'$, then for the objects and for the morphisms we have

$$HA = h^* : X^* \times Q \to V(Q \times Y^*),$$

$$H(\varphi_X, \varphi_Q, \varphi_Y) = (\varphi_{X^*}, \varphi_Q, \varphi_{Y^*}).$$

It can be easily verified that $H(\varphi_X, \varphi_Q, \varphi_Y)$ is a morphism of behaviours. The restriction of H over the subcategory $\overline{A}^S$ defines the functor $\overline{H} : \overline{A}^S \to \overline{B}^S$; it is quite obvious that this will be a functor : it follows from the condition $\varphi = (\varphi_X, \varphi_Q, \varphi_Y) \in \overline{A}^S$ that $\varphi_Q \in M^e$ and having in mind the construction of $\overline{B}^S$ we obtain $H(\varphi_X, \varphi_Q, \varphi_Y) \in \overline{B}^S$.

If the X-reduction of S-automaton $A = (X, Q, Y, h)$ is considered, then it is possible to build up one more category connected with the two constructed above. For the S-automaton A we introduce the relation r, which is defined for the elements of X :

$x_1 \, r \, x_2 \Leftrightarrow h_{(x_1,q)} = h_{(x_2,q)}$ for every $q \in Q$, where $x_1, x_2 \in X$; it could be easily proved, that r is a relation of equivalence, defined for the elements of X. Let X/r be the factorset of X by r and let h_r be the respective mapping, defined so that $h_r([x]_r, q) = h(x, q)$.

Definition 4. The S-automaton $A_r = (X/r, Q, Y, h_r)$ is a *strong X-reduced S-automaton* belonging to the S-automaton A.

Directly from this definition follows :

Proposition 2. *For each stochastic automaton there exist a strong X-reduced S-automaton, unique up to isomorphism.*

From the Definition 4 and Proposition 2 it follows that the strong X-reduced S-automata define a subcategory A_r^S of A^S, which we call *Category of strong X-reduced S-automata.* This will give the opportunity to build up a new functor $P : B^S \to A^S$ as follows

$$Ph^* = A_r = (X/r, Q, Y, h_r), \quad P(\varphi_{X^*}, \varphi_Q, \varphi_{Y^*}) = (\varphi_X, \varphi_Q, \varphi_Y)$$

where $h^* \in B_0^S$ and $(\varphi_{X^*}, \varphi_Q, \varphi_{Y^*}) \in B^S$.

From the commutativity of the following diagram

$$\begin{array}{ccc} X/r \times Q & \xrightarrow{\quad h_r \quad} & V(Q \times Y) \\ \varphi_X \times \varphi_Q \Big\downarrow & & \Big\downarrow V(\varphi_Q \times \varphi_Y) \\ X'/r' \times Q' & \xrightarrow{\quad h_r' \quad} & V(Q' \times Y') \end{array}$$

it follows that $P(\varphi_{X^*}, \varphi_Q, \varphi_{Y^*}) \in A^S$. The verification results from the fact, that the relation r', defined for each $q' \in Q'$ is compatible with r and that the condition $\varphi_X \times \varphi_Q : X/r \times Q \to X'/r' \times Q'$ is reduced to the necessity of φ_Q being epimorphism ; the last will be always true, as $Ph^* \in \overline{A}_r \cup \overline{A}^S$.

2. Universal Properties of the Functors H and P. Some auxiliary results will be given at the beginning and as there is no principle difficulty in the proves, they will be just sketched out in the text.

Lemma 1. *Let $A = (X, Q, Y, h) \in A_0^S$. The mapping $\eta_A : A \to PHA$ is an AS-morphism and defines a natural transformation $\eta : I_A^{-S} \to PH$.*

Proof. According to the construction of the functor H we have $HA = h^*$ for every $A \in A_0^S$. From the definition of P we obtain, that $PHA = (X/r, Q, Y, h_r)$ for the morphism of behaviours h^*. Then it follows that $\eta_A : A \to PHA$ is given by the ordered triple of mappings $\eta_A = (\varphi_X, i_Q, i_Y)$, where $\varphi_X : X \to X/r$, $i_Q : Q \to Q \in M^i$ and $i_Y : Y \to Y \in M^i$. Now it is easy to show the commutativity of the diagram

$$\begin{array}{ccc} X \times Q & \xrightarrow{\quad h \quad} & V(Q \times Y) \\ \varphi_X \times i_Q \Big\downarrow & & \Big\downarrow V(i_Q \times i_Y). \\ X/r \times Q & \xrightarrow{\quad h_r \quad} & V(Q \times Y) \end{array}$$

The second part of the Lemma comes by the commutativity of the following diagrams, equivalent between themselves.

$$\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[d, "\varphi"'] & PHA \arrow[d, "PH\varphi"] \\
A' \arrow[r, "\eta_{A'}"] & PHA
\end{tikzcd}$$

$$\begin{tikzcd}
X \times Q \arrow[rr, "h_A"] \arrow[dd, "\varphi_X \times \varphi_Q"'] \arrow[dr, "\varphi_r \times i_Q"] & & V(Q \times Y) \arrow[dd, "V(\varphi_Q \times \varphi_Y)"] \\
& X/r \times Q \arrow[ur, "h_{A_r}"'] \arrow[d, "\varphi_{x/r} \times \varphi_Q"'] & \\
X' \times Q' \arrow[rr, "h'_{A'}"] \arrow[dr, "\varphi_{r'} \times i_{Q'}"'] & & V(Q' \times Y') \\
& X'/r' \times Q' \arrow[ur, "h'_{A'_r}"'] &
\end{tikzcd}$$

L e m m a 2. *Let $h^* \in B_0^S$. The mapping $\varepsilon_{h^*} = (\varphi_{X^*}, i_Q, i_Y) : h_r^* \to h^*$ defines a morphism of behaviours $\varepsilon_{h^*} : HPh^* \to h^*$ and a natural transformation $\varepsilon : HP \to I_{\overline{B}^S}$.*

P r o o f. From the existence of an embedding $\varphi_{X^*} : X^*/r \to X^*$ and from the commutativity of the following diagram

$$\begin{tikzcd}
X^*/r \times Q \arrow[r, "h_r^*"] \arrow[d, "\varphi_{X^*} \times i_Q"'] & V(Q \times Y^*) \arrow[d, "V(i_Q \times i_{Y^*})"] \\
X^* \times Q \arrow[r, "h^*"] & V(Q \times Y^*)
\end{tikzcd}$$

it follows that $\varepsilon_{h^*} = (\varphi_{X^*}, i_Q, i_{Y^*})$ defines a morphism of behaviours $HPh^* \to h^*$. The second part of the Lemma results from the commutativity of the two equivalent diagrams

$$\begin{tikzcd}
HPh^* \arrow[r, "\varepsilon_h^*"] \arrow[d, "HP\varphi"'] & h^* \arrow[d] \\
HPh'^* \arrow[r, "\varepsilon_{h'^*}"] & h'^*
\end{tikzcd}$$

$$\begin{tikzcd}
X^*/r \times Q \arrow[rr, "h_r^*"] \arrow[dd, "\varphi_{X^*}/_r \times \varphi_Q"'] \arrow[dr] & & V(Q \times Y^*) \arrow[dd, "V(\varphi_Q \times \varphi_{Y^*})"] \\
& X^* \times Q \arrow[ur] \arrow[d, "h'_{r'}"] & \\
X'^*/r' \times Q' \arrow[rr, "h'^*_r"] \arrow[dr] & & V(Q' \times Y'^*) \\
& X'^* \times Q' \arrow[ur, "h'^*"'] & \\
& \varphi_{X^*} \times \varphi_Q &
\end{tikzcd}$$

P r o p o s i t i o n 3. *The ordered quadruple* $<\overline{H}, P, \eta, \varepsilon>$ *defines an adjunction* $\overline{A}^S \to \overline{B}^S$.

P r o o f. Given the diagrams

$$\begin{tikzcd} & HPHA \arrow[dr, "\varepsilon H_A"] & \\ HA \arrow[ur, "H\eta_A"] \arrow[rr, equal] & & HA \end{tikzcd} \qquad \begin{tikzcd} & HPPh^* \arrow[dr, "P\varepsilon h^*"] & \\ Ph^* \arrow[ur, "\eta Ph^*"] \arrow[rr, equal] & & Ph^* \end{tikzcd}$$

The commutativity of the first one follows from the relations

$$\varepsilon HA = \varepsilon_{HA} = \varepsilon_{h^*} : HPh^* \to h^*, \quad \varepsilon_{h^*} : h^*_r \to h^*, \text{ and}$$

$$H\eta_A = H(\eta_A : A \to PHA) : HA \to HA_r,$$

which obviously give $\varepsilon HA \circ H\eta_A = 1_h = 1_{HA}$; the commutativity of the second diagram follows from the relations:

$$\varepsilon h^* = \varepsilon_{h^*} : HPh^* \to h^*, \quad P\varepsilon h^* : Ph^*_r \to Ph^*, \quad P\varepsilon h^* : A_r \to A_r,$$

$$Ph^* = A_r, \quad \eta Ph^* : A_r \to PHA_r, \quad Ph^* : A_r \to A_r,$$

which also gives: $P\varepsilon h^* \circ \eta Ph^* = 1_{A_r} = 1_{Ph^*}$.

Considering the two Lemmas and the last proposition, we reach to the following basic result:

T h e o r e m. *The category* $\overline{A}_r$ *is a reflexive subcategory of* $\overline{A}^S$.

P r o o f. The statement of the Theorem might be expressed in the following way: for each automaton of $\overline{A}^S$ there exists an X-reduced S-automaton RA and an AS-morphism $\eta_A : A \to RA \in \overline{A}^S$, so that every AS-morphism $g : A \to A' \in \overline{A}^S$ is factorized by η_A for a unique morphism $k : RA \to A' \in \overline{A}_r$, i.e. the following diagram is commutative

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "g"'] & RA \arrow[d, dotted, "k"] \\ & A' \end{tikzcd}$$

Let $P(\overline{B}^S)$ be an image of the category $\overline{B}^S$ through P in $\overline{A}^S$; it is a reflexive subcategory of A^S, as for every $A \in \overline{A}^S_0$, the AS-morphism $\eta_A : A \to PHA$ is an universal arrow from P to A: the last is due to the fact, that $H \dashv P$, each adjunction defines a natural transformation $\eta : I_{\overline{A}^S} \to PH$ and that for each object $A \in A^S_0$ the arrow η_A is universal from P to A. We have to show now that the category $P(\overline{B}^S)$ is equivalent to $\overline{A}_r$; for the purpose let us consider $h, h' \in \overline{A}^S_0$ and the AS-morphism $g : Ph \to Ph' \in \overline{A}_r$, which gives the commutative diagram:

$$\begin{tikzcd}
X/r \times Q \arrow[r, "h"] \arrow[d, "\varphi_{X/r} \times \varphi_Q"'] & V(Q \times Y) \arrow[d, "V(\varphi_Q \times \varphi_Y)"] \\
X'/r' \times Q' \arrow[r, "h'"] & V(Q' \times Y') \\
X^*/r \times Q \arrow[r, "h^*"] \arrow[d, "\varphi_{X^*/r} \times \varphi_Q"'] & V(Q \times Y^*) \arrow[d, "V(\varphi_Q \times \varphi_{Y^*})."] \\
X'^*/r' \times Q' \arrow[r, "h'^*"] & V(Q' \times Y'^*)
\end{tikzcd}$$

Then the functor $P : \overline{B}^S \to \overline{A}^S$ is full and considered as a functor of embeding — faitfull; according to [4] each object $A \in A_0^S$ is isomorphic to Ph for a certain object $h \in \overline{B}_0^S$; the arrow $\eta_A : A \to PHA$ is an isomorphism for every $A \in A_{r0}$, as it is an ordered triple $\eta_A = (\varphi_{X/r}, i_Q, i_Y)$ and $\varphi_{X/r}$ is an isomorphism by the construction of P. Therefore $P(\overline{B}^S)$ is a category, equivalent to $\overline{A}_r$, but $P(\overline{B}^S)$ is a reflexive subcategory of $\overline{A}^S$ and heance $\overline{A}_r$ is a reflexive subcategory of $\overline{A}^S$.

This results is valide also for automata without an output but $V(Q \times \times Y)$ has to be replaced by VQ. The Theorem will be true also for Moore-type S-automata, as well as for deterministic automata [3].

3. Some Other Problems. By the so proved in this paper Theorem, the decision of the problem for a class of deterministic automata [3] is transfered to the case of strong X-reduction of S-automata. This type of reduction is only one of the possibilities and does not solve the problem as a whole.

Similar results are obtained for the Y-reduction and for the Q-reduction; they are interesting due to both their content and the explored technique. They will be an object of a later announcement.

Much more difficult is the general question for complete reduction of S-automata, i.e. simultaneously by X, Q and Y.

There are some principle difficulties in the case of stochastic automata and there appear two questions: under what conditions for the automaton it could be completely reduced, i.e. is it possible to describe a subcategory of A^S, whose objects are completely reduceable automata; what is the connection of the partial reductions (by X, by Q, by Y) with the complete reduction.

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Center of Applied Mathematics,
Sofia, Bulgaria

REFERENCES

1. Ehrig H., Kiermeir K. D., Kreowski H. J., Kühnel W., *Universal Theory of Automata. A Categorical Approach.* B. G. Teubner, Stuttgart, 1974.

2. Ehrig H., Kreowski H. J., Pfender M., *Kategorielle Theorie der Reduction, Minimierung und Äquivalenz von Automaten.* Math. Nachr., 59, *1 — 6*, 105 — — 124 (1974).
3. Goguen J., *Realization is Universal.* Math. Syst. Theory, 6, *4* (1972).
4. Mac-Lane S., *Categories for the Working Mathematician.* Springer Verlag, New York — — Heidelberg — Berlin, 1971.
5. Starke P., *Abstract Automata.* North-Holland Publ. Comp., Amsterdam —London, 1972.
6. Topenčarov V., Peeva K., *Algebraic Theory of Stochastic Automata. A Categorical Approach* (to appear).

O SUBCATEGORIE REFLEXIVĂ A CATEGORIEI AUTOMATELOR STOCHASTICE

(Rezumat)

Se construiesc categorii de automate stochastice şi de acţiuni de automate stochastice. Pentru o pereche de functori convenabili construiţi între aceste categorii se demonstrează proprietăţi de adjuncţie. Se dovedeşte că categoria de automat strict X-redus, asociat unei subcategorii bine precizate a categoriei automatelor stochastice este o subcategorie reflexivă.

Rezultatele obţinute sînt valabile de asemenea atît pentru automatele stochastice cît şi pentru automatele deterministe.

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SOME ELEMENTARY PROPERTIES OF THE MAPPINGS OF TOPOLOGICAL SPACES

BY

GH. COSTOVICI

1. In this Note we show from properties of the mappings of topological spaces and from properties of the compositions of these mappings with other result properties for the latter. Let X_i, $i \in I$ be a set of topological spaces with the topologies t_i, $i \in I$, respectively, and let $f_i : X_i \to X$ and $g_i : X \to X_i$, $i \in I$, be mappings. By t_{iq} (t_{ir}) we denote the topology on X defined in the following way : D is open in t_{iq} (t_{ir}) if and only if $f_i^{-1}(D)$ is open in t_i (exists D_i open in t_i so that $D = g_i^{-1}(D_i)$). They are called, respectively, the quotient topology relative to f_i and t_i, and, the reciprocical image of t_i by g_i. Also, on X, relative to all f_i. can by introduced the topologies o (from open) and f (from the french for closed) in the following way : D (F) is open (closed) in o (f) if and only if F (D) is arbitrary union (intersection) of finite intersections (unions) of $f_i(D_i)$ $(f_i(F_i))$ where i runs in I and D_i (F_i) is open (closed) in t_i. For every i, f_i is open (closed) relative to o (f) respectively. We recall that the set of the topologies defined on a set is complete lattice by the order $u \leqslant v$ if and only if every open in u is open in v. Then, on X, we have the following topologies : $\inf_i t_{iq} = I_q$, $\sup_i t_{iq} = S_q$, $\inf_i t_{ir} = I_r$, $\sup_i t_{ir} = S_r$, and where D is open in I_q (S_q) (I_r) (S_r) if and only if $f_i^{-1}(D)$ is open for every $i \in I$ (D is arbitrary union of finite intersections of D_i where i runs in I and $f_i^{-1}(D_i)$ is open in t_i) (for every i exists D_i open in t_i so that $D = g_i^{-1}(D_i)$) (D is arbitrary union of finite intersections of $g_i^{-1}(D_i)$ where i runs in I and D_i is open in t_i).

Every f_i (g_i) is continuous relative to t_i and I_q (relative to S_r and t_i), and if t' (t'') is another topology on X with respect to whom every f_i (g_i) is continuous, then, we have $t' \leqslant I_q \leqslant t_{iq} \leqslant S_q$ $(I_r \leqslant t_{ir} \leqslant S_r \leqslant t'')$. Generally, $t_{iqr} \leqslant t_i$ $(t_i \leqslant t_{irq})$ and if f_i (g_i) is injection (surjection) we have $t_{iqr} = t_i$ $(t_i = t_{irq})$ (for proof one uses $f_i^{-1}(f_i(D_i)) = D_i$ if f_i is injection and $g_i(g_i^{-1}(D_i) = D_i$ if g_i is surjection).

If g_i is injection and t_i is T_0-topology (distinct points do not have identical closures), then, t_{ir} is T_0 - topology (easy to prove).

If $h : I \to \bigcup_i X_i$, put $(x_i) = h(I)$ if and only if $h(i) = x_i \in X_i$ for every $i \in I$ and, also, denote $\Pi X_i = \{(x_i)\}$. The mapping $p_i : \Pi X_i \to X_i$ so

that $p_i(x_i) = x_i$, is called the projection of the product ΠX_i on X_i. One considers ΠX_i equiped with the topology S_r relative to all p_i and t_i and it is called the product topology. If X is topological space and $k \in I$, then, the mapping $p : \Pi X_i \to X$ is k-continuous (k-open) (k-closed) (k-injection, (k-surjection) if and only of for every $(x_i ; i \neq k)$, $x_i \in X_i$, $i \in I$, the mapping $p_{(x_i ; i \neq k)} : X_k \to X$ where $P_{(x_i ; i \neq k)}(x'_k) = P(x_i ; x_k = x'_k)$ is continuous (open) (closed) (injection) (surjection), respectively.

2. Theorem 1. *If X_i, $i = \overline{1, 3}$, are topological spaces with the topologies t_i, $i = \overline{1, 3}$, respectively, Y_k, $k \in K$, are topological spaces with the topologies T_k, $k \in K$, respectively, $f_j : X_j \to X_{j+1}$, $j = \overline{1, 2}$, $g_k : X_2 \to Y_k$, $k \in K$, $h_k : Y_k \to X_2$, $k \in K$, then, we have:*

(1) *$f_2 f_1$ continuous, f_1 open or closed, f_1 surjection $\Rightarrow f_2$ continuous;*
(2) *$t_2 = 0$ or $t_2 = f$, relative to all h_k, exists h_{k_0} surjection and $f_2 h_{k_0}$ continuous $\Rightarrow f_2$ continuous;*
(3) *$t_2 = S_q$ relative to all h_k and T_k, exists $f_2 h_{k_0}$ continuous $\Rightarrow f_2$ continuous;*
(4) *$t_2 = I_q$ relative to all h_k and $T_q \Rightarrow (f_2 h_q$ continuous for every $k \Leftrightarrow f_2$ continuous);*
(5) *$f_2 f_1$ continuous, f_2 open or closed, f_2 injection $\Rightarrow f_1$ continuous;*
(6) *$t_3 = 0$ or $t_3 = f$, relative to f_2, $f_2 f_1$ continuous, f_2 injection $\Rightarrow f_1$ continuous;*
(7) *$t_2 = I_r$ relative to all g_k and T_k, exists $g_{k_0} f_1$ continuous $\Rightarrow f_1$ continuous;*
(8) *$t_2 = S_r$ relative to all g_k and $T_k \Rightarrow (g_k f_1$ continuous for every $k \Leftrightarrow f_1$ continuous);*
(9) *$f_2 f_1$ open (closed), f_2 continuous injection $\Rightarrow f_1$ open (closed);*
(10) *$t_2 = S_r$ relative to all g_k and T_{k_0}, exists g_{k_0} injection, $g_{k_0} f_1$ open (closed) $\Rightarrow f_1$ open (closed);*
(11) *$f_2 f_1$ open (closed), f_1 continuous surjection $\Rightarrow f_2$ open (closed);*
(12) *$t_2 = I_\varsigma$ relative to all h_k and T_k, exists h_{k_0} surjection, $f_2 h_{k_0}$ open (closed) $\Rightarrow f_2$ open (closed).*

Proof. We check only (1) the rest following same way. Let f_1 be open and let D_3 be t_3-open. Then, $(f_2 f_1)^{-1}(D_3) = f_1^{-1}(f_2^{-1}(D_3)) = D_1$ is t_1-open. Hence, $f_2^{-1}(D_3) = f_1(D_1)$ is t_2-open. If f_1 is closed and F_3 is t_3-closed, then, $(f_2 f_1)^{-1}(F_3) = f_1^{-1}(f_2^{-1}(F_3)) = F_1$ is t_1-closed, and then, $f_2^{-1}(F_3) = f_1(F_1)$ is t_2-closed.

Remark. The point (3) or (4) when K contains only one element, appears in [1, p. 97] and in Proposition 4.1 [2].

Theorem 2. *If A, X, A_i, X_i, $i \in I$ are topological spaces, X_i with the topology t_i, $i \in I$, exists $k \in I$ so that $f_i : A_i \to X_i$ are surjections for every $i \in (I - \{k\})$ and the following square commutative:*

$$\begin{tikzcd} \Pi X_i \arrow[r, "p"'] & X \\ \Pi A \arrow[u, "(f_i)"] \arrow[r, "q"'] & A \arrow[u, "f"'] \end{tikzcd}$$

then, we have:

(1) *f_k open or closed, f_k surjection, f continuous, q k-continuous $\Rightarrow$ $\Rightarrow p$ k-continuous;*

(2) f_k *surjection*, $t_k = 0$ *or* $t_k = f$, *relative to* f_k, f *continuous*, q *k-continuous* $\Rightarrow p$ *k-continuous*;

(3) $t_k = t_{kq}$ *relative to* f_k, f *continuous*, q *k-continuous* $\Rightarrow p$ *k-continuous*;

(4) f *continuous*, q *k-continuous*, p *k-open or k-closed*, p *k-injection* $\Rightarrow f_k$ *continuous*;

(5) *exists* $(a_i;\ i \neq k)$, $a_i \in A_i$, $i \in (I - \{k\})$ *so that* $q_{(a_i;\ i \neq k)}$ *continuous*, $p_{(x_i = f_i(a_i)\ :\ i \neq k)}$ *injection, the topology on* X *equals the topology* o *or* f *relative to* $p_{(x_i = f_i(a_i)\ :\ i \neq k)}$, f *continuous* $\Rightarrow f_k$ *continuous*;

(6) $t_k = I_r$ *relative to all* $p_{(x_i\ :\ i \neq k)}$, f *continuous, exists* $q_{(a_i^0\ :\ i \neq k)}$ *continuous* $\Rightarrow f_k$ *continuous*;

(7) $t_k = S$, *relative to all* $p_{(x_i\ :\ i \neq k)}$, f *continuous*, q *k-continuous* $\Rightarrow f_k$ *continuous*;

(8) *exists* $(a_i;\ i \neq k)$, $a_i \in A_i$, $i \in (I - \{k\})$ *so that* $q_{(a_i\ :\ i \neq k)}$ *is closed (open)*, f *closed (open)*, $t_k = S_r$ *relative to all* $p_{(x_i\ :\ i \neq k)}$, $p_{(x_i = f_i(a_i)\ :\ i \neq k)}$ *injection* $\Rightarrow f_k$ *closed (open), respectively*;

(9) f_k *surjection*, $t_k = t_{kq}$ *relative to* f_k *and the topology on* A_k, f *open (closed)*, q *k-open (k-closed)* $\Rightarrow p$ *k-open (k-closed)*.

Proof. Given $(x_i;\ i \neq k)$, exists $(a_i;\ i \neq k)$ so that $f_i(a_i) = x_i$, $i \neq k$. The following square is commutative:

$$\begin{tikzcd} X_k \arrow[r, "p_{(x_i\ :\ i \neq k)}"] & X \\ A_k \arrow[u, "f_k"] \arrow[r, "q_{(a_i\ :\ i \neq k)}"'] & A \arrow[u, "f"'] \end{tikzcd}$$

Theorem 2 follows then from Theorem 1.

Remark. The point (3) from Theorem 2 is almost Proposition 4.1 from [2].

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Polytechnic Institute, Jassy,
Department of Mathematics

REFERENCES

1. Costinescu O., Amihăesei C., Bârsan T., *Topologie generală. Probleme.* Bucureşti, 1974.
2. Lawson J. D., *Joint Continuity in Semitopological Semigroups.* Illinois J. Math., 18, 2, 275 – 285 (1974).

UNELE PROPRIETĂŢI ELEMENTARE ALE APLICAŢIILOR DE SPAŢII TOPOLOGICE

(Rezumat)

Din proprietăţi ale aplicaţiilor de spaţii topologice (cu topologii mai mult sau mai puţin particulare), şi ale compuselor acestora cu altele la stînga sau la dreapta, se evidenţiază proprietăţi pentru acestea din urmă.

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O EXTINDERE A NOŢIUNII DE TRANSPORT PRIN PARALELISM ÎN S_4

DE

ŞTEFANIA RUSCIOR

În această Notă se extinde noţiunea de transport prin paralelism în sens Levi-Civita din S_3. Şcoala de geometrie din Iaşi a adus contribuţii importante în această problemă [1]...[3]. Problema a fost reluată în lucrările [4], [5].

Se ştie că pentru a defini transportul prin paralelism de-a lungul unei curbe (γ) de pe o suprafaţă (S) se consideră un cîmp de vectori $\overline{V}(u)$ de-a lungul curbei şi variaţia vectorului $\delta\overline{V}$ din planul tangent (T) dus suprafeţei într-un punct curent al curbei (γ). Dacă $\delta\overline{V} = 0$ şi planul tangent descrie o suprafaţă desfăşurabilă, atunci se defineşte un transport prin paralelism al cîmpului $\overline{V}$ de-a lungul curbei (γ).

Fie în (S_4) o hipersuprafaţă (Σ) şi pe ea o curbă. Considerăm un cîmp de vectori $\overline{V}(u)$ de-a lungul curbei (γ). Impunem condiţii de regularitate funcţiilor care determină hipersuprafaţa (Σ), curba (γ) şi cîmpul de vectori. În aceste condiţii în fiecare punct nesingular al hipersuprafeţei se determină un hiperplan tangent (H).

Fie M un punct al cubei (γ) şi hiperplanul tangent (H) dus în acest punct hipersuprafeţei. În punctul M se determină vectorul $\overline{V}$ al cîmpului considerat, iar în punctul M_1 vectorul $\overline{V}_1$ al aceluiaşi cîmp. Hiperplanul (H) intersectează vectorul $\overline{V}$ în punctul A şi vectorul $\overline{V}_1$ în punctul A_1 Se determină astfel $\overline{AA_1}$ conţinut în (H) pe care-l vom numi variaţia lui V în hiperplanul tangent de-a lungul curbei (γ). Fie $\delta\overline{V}$ acest vector. Vom spune că avem un transport prin paralelism al cîmpului $\overline{V}$ de-a lungul curbei (γ) dacă $\delta\overline{V} = 0$ şi hiperplanul tangent (H) dus de-a lungul curbei descrie o hipersuprafaţă desfăşurabilă. Astfel am extins noţiunea de transport prin paralelism în S_4.

Ne propunem să determinăm condiţiile analitice ca un cîmp de vectori să se transporte paralel de-a lungul unei curbe aşezată pe o hipersuprafaţă riglată dată. Ecuaţia hipersuprafeţei este dată vectorial prin

$$\overline{r} = \overline{r}(u, v, w) = \overline{\rho}(u, v) + w\,\overline{\alpha}(u, v). \tag{1}$$

O curbă oarecare (γ) aşezată pe ea poate fi luată ca linie parametrică a hipersuprafeţei caracterizată prin relaţiile $v = \text{const.}$ şi $w = \text{const.}$, fără a restrînge generalitatea problemei. Hiperplanul tangent dus într-un punct curent M al curbei este dat de ecuaţia

$$[\bar{R} - \bar{\rho},\ \bar{\rho}_u + w\bar{\alpha}_u,\ \bar{\rho}_v + w\bar{\alpha}_v,\ \bar{\alpha}] = 0. \tag{2}$$

Cîmpul de vectori considerat $\bar{V}$ se poate scrie în aceste condiţii sub forma

$$\bar{V} = \bar{r} + A\bar{e}_1 + B\bar{e}_2 + C\bar{e}_3 + D\bar{e}_4, \tag{3}$$

unde $\bar{e}_i$ sînt versorii fundamentali şi A, B, C sînt scalari ce depind de arcul u de pe curba (γ). Variaţia vectorului $\bar{V}$ de-a lungul curbei (γ) este conţinută în hiperplanul tangent (H). Transportul prin paralelism este caracterizat prin relaţia

$$\delta\bar{V} = \mathrm{d}\bar{r} + \mathrm{d}A\bar{e}_1 + \mathrm{d}B\bar{e}_2 + \mathrm{d}C\bar{e}_3 + \mathrm{d}D\bar{e}_4. \tag{4}$$

Impunem şi condiţia ca hiperplanul tangent dus de-a lungul curbei (γ) să formeze o hipersuprafaţă desfăşurabilă şi găsim condiţiile de transport prin paralelism. Hiperplanul tangent formează o hipersuprafaţă desfăşurabilă de-a lungul unei curbe, pe hipersuprafeţele particulare (V_2^d).

După natura secţiunii improprii Σ_i avem mai multe cazuri:

a) Σ_i este o suprafaţă desfăşurabilă dacă $\bar{\alpha} = \bar{a}(u) + v\bar{a}'(u)$.

b) Σ_i este un con dacă $\bar{\alpha} = \bar{a} + v\bar{b}(u)$, cu $\bar{a} = \text{const.}$

c) Σ_i este un cilindru dacă $\bar{\alpha} = \bar{a}(u) + v\bar{b}$, cu $\bar{b} = \text{const.}$

d) Σ_i este o curbă dacă $\bar{\alpha} = \bar{a}(u)$.

e) Σ_i este un punct dacă $\bar{\alpha} = \bar{a}$, cu $\bar{a} = \text{const.}$

Ne propunem să exprimăm aceste condiţii pe hipersuprafeţe riglate particulare de tipul (V_2^d) cu două puncte singulare pe o generatoare generică cu secţiunea improprie de dimensiune d.

Din aceste hipersuprafeţe fac parte cele obţinute din mişcarea unui solid rigid fără punct fix în S_4 [6], [7], [8] cu secţiunea improprie o curbă.

Să considerăm ecuaţia hipersuprafeţei

$$\bar{r} = \bar{\rho}(u) + v\,\bar{a}(u) + w\bar{b}(u). \tag{5}$$

Curba (γ) este dată pe hipersuprafaţă prin condiţiile $v = \text{const.}$ şi $w = \text{const.}$ şi are ecuaţia $\bar{\rho} = \bar{\rho}(u)$.

Hiperplanul tangent (H) are ecuaţia:

$$[\bar{R} - \bar{\rho},\ \bar{\rho}' + v\bar{a}' + w\bar{b}',\ \bar{a},\ \bar{b}] = 0. \tag{6}$$

Acest hiperplan descrie o hipersuprafaţă riglată de-a lungul curbei dacă vectorii ($\bar{a}$, $\bar{b}$, $\bar{a}'$, $\bar{b}'$, $\bar{\rho}'$) sînt în acelaşi hiperplan. Matricea corespunzătoare componentelor acestor vectori are rangul 3, prin urmare trei dintre aceşti vectori sînt independenţi.

Cîmpul $\overline{V}$ de-a lungul curbei (γ) are ecuaţia

$$\overline{V} = \overline{\rho} + A(u)\,\overline{\rho}' + B(u)\,\overline{a} + C(u)\,\overline{b}, \tag{7}$$

unde A, B, C sînt scalari oarecari, funcţie de arcul curbei. Deplasarea vectorului de-a lungul curbei din hiperplaunl tangent este

$$\delta\overline{V} = (\overline{\rho}' + A'\,\overline{\rho}' + A\overline{\rho}'' + B'\,\overline{a} + B\overline{a}' + C'\,\overline{b} + C\overline{b}')\,du. \tag{8}$$

Transportul prin paralelism de-a lungul curbei (γ) rezultă din $\delta\overline{V} = 0$. Aceasta se exprimă analitic simplu dacă alegem un sistem local convenabil şi anume : originea în punctul M iar versorii fundamentali : $\overline{a}(1, 0, 0, 0)$, $\overline{\rho}'(0, 0, 1, 0)$, $\overline{b}(0, 1, 0, 0)$, $\overline{\rho}''(0, 0, 0, 1)$. Vectorii $\overline{a}'$, $\overline{b}'$ care depind linear de vectorii $\overline{a}$, $\overline{b}$ şi $\overline{\rho}'$, $\overline{\rho}''$ au componentele $\overline{a}'(a_1, a_2, a_3, 0)$, $\overline{b}'(b_1, b_2, b_3, 0)$.

Obţinem sistemul de condiţii pentru transportul prin paralelism :

$$\begin{aligned} &B' + Ba_1 + Cb_1 = 0,\\ &C' + Ba_2 + Cb_2 = 0,\\ &1 + A' + Ba_3 + Cb_3 = 0,\\ &A = 0. \end{aligned} \tag{9}$$

Din ultima relaţie a sistemului (9) rezultă că pentru a avea un transport prin paralelism, vectorul $\overline{V}$ are forma :

$$\overline{V} = \overline{\rho} + B\overline{a} + C\overline{b}.$$

Rezultă că versorul său se află pe o direcţie aşezată în planul de bază (Π), determinat de vectorii $\overline{a}$ şi $\overline{b}$.

Din primele două ecuaţii diferenţiale din (9) rezultă că funcţiile B şi C sînt soluţii ale sistemului şi conţin două constante arbitrare. Deoarece aceste soluţii trebuie să satisfacă şi ecuaţia a treia, între constantele arbitrare introduse apare o relaţie. Prin urmare problema transportului prin paralelism pe hipersuprafeţe riglate de tip V_2^d depinde de o singură constantă arbitrară. Geometric aceasta exprimă faptul că direcţia cîmpului $\overline{V}$ de vectori este determinată în planul de bază (Π), dacă se cunoaşte unghiul pe care-l face cu una din direcţiile $\overline{a}$ sau $\overline{b}$.

Am studiat în continuare modul cum se transportă prin paralelism de-a lungul unei curbe (γ) aşezată pe o hipersuprafaţă un cîmp de vectori, dacă hipersuprafaţa (Σ) este pusă în corespondenţă prin hiperplane tangente paralele cu hipersuprafaţa (Σ^*) din aceeaşi categorie [6].

Folosind rezultatele obţinute prin corespondenţe prin hiperplane tangente paralele, cîmpul de vectori V se transformă în V^* de-a lungul curbei (γ^*) de pe (Σ^*) corespunzătoare lui (Σ).

Prin urmare dacă un cîmp de vectori se transportă paralel de-a lungul curbei (γ) transformarea prin hiperplane tangente paralele determină pe hipersuprafaţa (Σ^*) o curbă (γ^*) şi un cîmp V^* ce se transportă paralel de-a lungul curbei (γ^*) de pe (Σ^*).

Rezultă următoarea

T e o r e m ă. *Transportul prin paralelism de-a lungul unei curbe* (γ) *de pe o suprafaţă riglată* (Σ) *se păstrează pe o hipersuprafaţă* (Σ^*) *pusă în corespondenţă prin hiperplane tangente paralele de-a lungul unei curbe* (γ^*) *corespunzătoare lui* (γ).

Rezultatele obţinute pot fi extinse în spaţiul afin cu n dimensiuni pe varietăţi riglate care păstrează hiperplanul fix de-a lungul unei generatoare.

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Institutul politehnic, Iaşi,
Catedra de matematici

BIBLIOGRAFIE

1. M y l l e r Al., *Le parallélisme au sens Levi-Civita, dans un système des plans.* Ann. Sci. de l'Univ. de Jassy, XIII, *1 — 2*, 20 — 34 (1924).
2. M y l l e r Al., *Curbe şi suprafeţe paralele în sens larg.* St. şi cerc. Acad. R.P. R. Iaşi, V, *1 — 2*, 1 — 15 (1954).
3. M a y e r O., *Sur les surfaces réglées.* Ann. Sci. de l'Univ. de Jassy, Section Math. - - Physique - Chimie, XXVI, *1*, 299 — 308 (1940).
4. C o n s t a n t i n e s c u I. N., K e c s W., *Les propriétés du trièdre de Darboux concernant le transport par parallélisme.* Przeglad Naukowo Techniczny Akademii Gorniczo--Hutniozej, Krakov, Seria H, 15, *8*, 19 — 23 (1965).
5. C o n s t a n t i n e s c u I. N., K e c s W., Tagung über Grundlagen der Geometrie, Oberwolfach, 30 Mai — 4 Juni, 1966.
6. R u s c i o r Şt., *Corespondenţa prin hiperplane tangente paralele între două hipersuprafeţe riglate din S_4 afin.* An. Şt. Univ. „Al. I. Cuza" Iaşi, VIII, *2*, 369 — 394 (1962).
7. R u s c i o r Şt., *Sur une classification affine des hipersurfaces réglées dans E_4.* Bull. Acad. Roy. Belgique, Cl. sci. math., L, 309 — 314 (1964).
8. R u s c i o r Şt., *Caractérisation géométrique des variétés générées par des mouvements d'un solide rigide sans point fix dans S_4.* Bul. Inst. polit. Iaşi, XIII (XVII), *1—2* (1967).

SUR LE TRANSPORT PAR PARALLÉLISME DANS UN E_4 AFFINE

(Résumé)

Dans cette Note on obtient le résultat : la correspondance par hyperplans tangents parallèles qui conservent les génératrices des hypersurfaces réglées (Σ) et (Σ^*) transforme une courbe (γ) situeé sur (Σ), dans une courbe (γ^*) située sur (Σ^*), et le champ des vecteurs $\overline{V}$ qui se transportent per parallélisme le long de (γ) devient un champ de vecteurs $\overline{V}^*$ qui se transportent par parallélisme le long de (γ^*).

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CONEXIUNI FINSLERIENE CONFORM APROAPE SIMPLECTICE

DE

I. GHINEA

În această lucrare ne propunem să determinăm forma generală a conexiunilor finsleriene compatibile cu o structură conform aproape simplectică, plecînd de la definiţia conexiunii finsleriene, dată de M. Matsumoto [2], [3].

Notaţiile şi formulele utilizate sînt cele din [2], [3].

1. Fie M o varietate finsleriană, diferenţiabilă, de dimensiune $2n$. Să considerăm pe M o conexiune finsleriană $(N_j^i, F_{jk}^i\ C_{jk}^i)$ definită ca în [2], [3], împreună cu algoritmul de derivare covariantă în raport cu această conexiune, pentru cîmpurile de tensori din M. Astfel pentru un cîmp de tensori T_j^i se pot defini următoarele derivate covariante:

$$(1.1)\qquad \begin{cases} T^i_{j|k} = \dfrac{\delta T_j^i}{\delta x^k} + T_j^h F_{hk}^i - T_h^i F_{jk}^h, & h\text{-derivata covariantă}, \\[2ex] T^i_{j|k} = \dfrac{\partial T_j^i}{\partial y^k} + T_j^h C_{hk}^i - T_h^i C_{jk}^h, & v\text{-derivata covariantă}, \end{cases}$$

unde $\dfrac{\delta}{\delta x^k} = \dfrac{\partial}{\partial x^k} - N_k^j \dfrac{\partial}{\partial x^j}$.

O structură finsleriană conform aproape simplectică pe M este definită printr-un cîmp de pseudotensori $a_{ij}(x, y)$, (y fiind elementul de suport în punctul x) cu proprietăţile:

1) $a_{ij}(x, y)$ este definit pînă la un factor local pozitiv, arbitrar;

2) $\rho a_{ij}(x, y)$ este omogen de grad zero în raport cu elementul de suport, deci $\rho(x, y)$ este o funcţie omogenă de grad zero în raport cu y;

3) $a_{ij}(x,y) = -a_{ji}(x,y)$; 4) $\det a_{ij}(x,y) \neq 0, \quad \forall y \neq 0$.

Dacă ρa_{ij} este dat, atunci există cîmpul de pseudotensori $\dfrac{1}{\rho} a^{ij}$ cu proprietatea că

$$(1.2)\qquad a_{ij}\, a^{hj} = \delta_i^h, \qquad a_{ji}\, a^{jh} = \delta_i^h.$$

Asociem structurii conform aproape simplectice ρa_{ij} operatorii

$$(1.3)\qquad \theta_{kh}^{ij} = \frac{1}{2}(\delta_k^i \delta_h^j - a_{kh}\, a^{ji}), \qquad \theta_{kh}^{*ij} = \frac{1}{2}(\delta_k^i \delta_h^j + a_{kh}\, a^{ji}),$$

care sînt omogeni de grad zero în y, sînt invarianţi la transformările $a_{ij} \to \rho a_{ij}$ şi satisfac relaţiile

$$\theta + \theta^* = \delta\delta, \quad \theta\theta = \theta, \quad \theta\theta^* = 0, \quad \theta^*\theta = 0, \quad \theta^*\theta^* = \theta^*. \tag{1.4}$$

2. Ne propunem să căutăm conexiuni finsleriene $(N^i_j, F^i_{jk}, C^i_{jk})$ cu proprietatea

$$a_{ij/k} = 2\omega_k a_{ij}, \qquad a_{ij}/_k = 2\lambda_k a_{ij}, \tag{2.1}$$

unde $\omega_k = \omega_k(x, y)$ şi $\lambda_k = \lambda_k(x, y)$ sînt vectori covarianţi omogeni în y de grad 0 respectiv -1.

D e f i n i ţ i a 1. O conexiune finsleriană $(N^i_j, F^i_{jk}, C^i_{jk})$ se numeşte compatibilă cu structura conform aproape simplectică ρa_{ij} dacă verifică relaţiile (2.1). În acest caz vom spune că $(N^i_j, F^i_{jk}, C^i_{jk})$ este conexiune finsleriană conform aproape simplectică.

Prin derivarea covariantă a relaţiilor (1.2) se obţine

P r o p o z i ţ i a 1. *Dacă* $(N^i_j, F^i_{jk}, C^i_{jk})$ *este o conexiune finsleriană conform aproape simplectică atunci avem*

$$a^{ij}{}_{/k} = -2\omega_k a^{ij}, \qquad a^{ij}/_k = -2\lambda_k a^{ij}. \tag{2.2}$$

Dacă acum se derivează covariant relaţiile (1.3) rezultă

P r o p o z i ţ i a 2. *Operatorii* (1.3) *sînt covariant constanţi în raport cu orice conexiune finsleriană conform aproape simplectică* $(N^i_j, F^i_{jk}, C^i_{jk})$.

Pentru a găsi conexiunile finsleriene $(N^i_j, F^i_{jk}, C^i_{jk})$ conform aproape simplectice procedăm astfel: fie $(\mathring{N}^i_j, \mathring{F}^i_{jk}, \mathring{C}^i_{jk})$ o conexiune fixată pe M. Căutăm cîmpurile de tensori $A^i_j, B^i_{jk}, D^i_{jk}$ cu proprietatea că conexiunea $(N^i_j, F^i_{jk}, C^i_{jk})$ dată de

$$N^i_j = \mathring{N}^i_j + A^i_j, \qquad F^i_{jk} = \mathring{F}^i_{jk} + B^i_{jk}, \qquad C^i_{jk} = \mathring{C}^i_{jk} + D^i_{jk} \tag{2.3}$$

să verifice relaţiile (2.1). Dacă substituim (2.3) în (2.1) obţinem

$$\begin{cases} a_{hj} B^h_{ik} + a_{ih} B^h_{jk} = \mathring{\nabla}_k a_{ij} - 2\omega_k a_{ij} - A^h_k \dfrac{\partial a_{ij}}{\partial y^h}, \\[2ex] a_{hj} D^h_{ik} + a_{ih} D^h_{jk} = \mathring{\dot{\nabla}} a_{ij} - 2\lambda_q a_{ij}, \end{cases} \tag{2.4}$$

unde $\mathring{\nabla}_k a_{ij}$ şi $\mathring{\dot{\nabla}}_k a_{ij}$ sînt h-derivata, respectiv v-derivata covariantă a tensorului a_{ij} în raport cu conexiunea fixată $(\mathring{N}^i_j, \mathring{F}^i_{jk}, \mathring{C}^i_{jk})$.

Dacă înmulţim (2.4) cu $\frac{1}{2} a^{il}$ şi punem în evidenţă operatorul θ^{*li}_{hj} dat de (1.3) obţinem ecuaţiile tensoriale

$$\theta^{*li}_{hj} B^h_{ik} = \frac{1}{2} a^{il} \left(\mathring{\nabla}_k a_{ij} - 2\omega_k a_{ij} - A^h_k \frac{\partial a_{ij}}{\partial y^h} \right), \tag{2.5}$$

$$\theta^{*li}_{hj} D^h_{ik} = \frac{1}{2} a^{il} (\mathring{\dot{\nabla}} a_{ij} - 2\lambda_k a_{ij}). \tag{2.6}$$

Se verifică uşor că pentru aceste ecuaţii sînt îndeplinite condiţiile de compatibilitate. Rezultă că aceste ecuaţii au soluţii în raport cu B^h_{ik}, D^h_{ik} oricare ar fi tensorul A^h_k omogen de grad 1 în raport cu y, date de

$$B^h_{jk} = \frac{1}{2} a^{ph} \left(\overset{\circ}{\nabla}_k a_{pj} - 2\omega_k a_{pj} - A^q_k \frac{\partial a_{pj}}{\partial y^q} \right) + \theta^{ph}_{jq} U^q_{pk}, \tag{2.7}$$

$$D^h_{jk} = \frac{1}{2} a^{ph} (\overset{\circ}{\dot{\nabla}}_k a_{pj} - 2\lambda_k a_{pj}) + \theta^{ph}_{jq} V^q_{pk}, \tag{2.8}$$

unde U^q_{pk} şi V^q_{pk} sînt tensori arbitrari, omogeni de grad 0 respectiv -1 în raport cu y.

Dacă notăm $P^h_{jk} = \theta^{ph}_{jq} U^q_{pk}$ şi $Q^h_{jk} = \theta^{ph}_{jq} V^q_{pk}$ rezultă că P^h_{jk} şi Q^h_{jk} sînt tensori hibrizi în indicii h şi j.

Acum ţinînd seama în (2.3) de (2.7) şi (2.8) precum şi de notaţiile precedente obţinem

$$\begin{cases} N^h_j = \overset{\circ}{N}{}^h_j + A^h_j, \\ F^h_{jk} = \overset{\circ}{F}{}^h_{jk} + \frac{1}{2} a^{ph} \left(\overset{\circ}{\nabla}_k a_{pj} - 2\omega_k a_{pj} - A^q_k \frac{\partial a_{pj}}{\partial y^q} \right) + P^h_{jk}, \\ C^h_{jk} = \overset{\circ}{C}{}^h_{jk} + \frac{1}{2} a^{ph} (\overset{\circ}{\dot{\nabla}}_k a_{pj} - 2\lambda_k a_{pj}) + Q^h_{jk}. \end{cases} \tag{2.9}$$

Rezultă

T e o r e m a 1. *Toate conexiunile finsleriene* $(N^h_j, F^h_{jk}, C^h_{jk})$ *conform aproape simplectice sînt date de formulele* (2.9) *în care* $(\overset{\circ}{N}{}^h_j, \overset{\circ}{F}{}^h_{jk}, \overset{\circ}{C}{}^h_{ik})$ *este o conexiune fixată,* $\overset{\circ}{\nabla}_k$ *şi* $\overset{\circ}{\dot{\nabla}}_k$ *sînt h-derivata şi v-derivata covariantă în raport cu această conexiune, iar* $A^h_j, P^h_{jk}, Q^h_{jk}$ *sînt tensori arbitrari omogeni de grad 0, 1, —1 în raport cu elementul de suport y, ultimii doi fiind hibrizi în indicii h şi j.*

3. Să considerăm acum drept conexiune fixată $(\overset{\circ}{N}{}^h_j, \overset{\circ}{F}{}^h_{jk}, \overset{\circ}{C}{}^h_{jk})$ o conexiune aproape simplectică a_{ij}, deci $\overset{\circ}{\nabla}_k a_{ij} = 0$ şi $\overset{\circ}{\dot{\nabla}}_k a_{ij} = 0$, [8]. Atunci formulele (2.9) devin

$$\begin{cases} N^h_j = \overset{\circ}{N}{}^h_j + A^h_j, \\ F^h_{jk} = \overset{\circ}{F}{}^h_{jk} - \delta^h_j \omega_k - \frac{1}{2} A^q_k a^{ph} \frac{\partial a_{pj}}{\partial y^q} + \theta^{ph}_{jq} U^q_{pk}, \\ C^h_{jk} = \overset{\circ}{C}{}^h_{jk} - \delta^h_j \lambda_k + \theta^{ph}_{jq} V^q_{pk}. \end{cases} \tag{3.1}$$

Rezultă astfel

T e o r e m a 2. *Dacă* $(\overset{\circ}{N}{}^h_j, \overset{\circ}{F}{}^h_{jk}, \overset{\circ}{C}{}^h_{jk})$ *este o conexiune finsleriană aproape simplectică a lui* a_{ij}, *atunci mulţimea conexiunilor* $(N^h_j, F^h_{jk}, C^h_{jk})$ *compatibile cu structura conform aproape simplectică* ϱa_{ij} *este dată de formulele* (3.1) *unde* A^h_j, U^q_{pk} *şi* V^q_{pk} *sînt tensori arbitrari pe M, omogeni de grad 0, 1, —1 respectiv în raport cu elementul de suport y.*

Întrucît în (3.1) $A^h_j, U^q_{pk}, V^q_{pk}$ sînt tensori arbitrari, rezultă că se pot impune diverse condiţii :

a) Să determinăm conexiunile finsleriene conform aproape simplectice cu proprietatea că N_j^h este fixat, deci

$$N_j^h = \mathring{N}_j^h, \quad F_{jk}^h = \mathring{F}_{jk}^h + B_{jk}^h, \quad C_{jk}^h = \mathring{C}_{jk}^h + D_{jk}^h. \tag{3.2}$$

Rezultă că $A_j^h = 0$ şi din (3.1) avem

Teorema 3. *Dacă* $(\mathring{N}_j^h, \mathring{F}_{jk}^h, \mathring{C}_{jk}^h)$ *este o conexiune finsleriană aproape simplectică, atunci toate conexiunile finsleriene* $(N_j^h, F_{jk}^h, C_{jk}^h)$ *conform aproape simplectice cu proprietatea* (3.2) *sînt date de*:

$$\left\{\begin{aligned} N_j^h &= \mathring{N}_j^h, \\ F_{jk}^h &= \mathring{F}_{jk}^h - \delta_j^h \omega_k + \theta_{jq}^{ph} U_{pk}^q, \\ C_{jk}^h &= \mathring{C}_{jk}^h - \delta_j^h \lambda_k + \theta_{jq}^{ph} V_{pk}^q. \end{aligned}\right. \tag{3.3}$$

b) Să considerăm acum cazul cînd

$$A_j^h = 0, \quad U_{pk}^q = 0, \quad V_{pk}^q = 0. \tag{3.4}$$

Ţinînd seama de (3.4) în (3.1) rezultă

Teorema 4. *Dacă* $(\mathring{N}_j^h, \mathring{F}_{jk}^h, \mathring{C}_{jk}^h)$ *este o conexiune finsleriană aproape simplectică atunci conexiunea* $(N_j^h, F_{jk}^h, C_{jk}^h)$ *dată de*

$$N_j^h = \mathring{N}_j^h, \quad F_{jk}^h = \mathring{F}_{jk}^h - \delta_j^h \omega_k, \quad C_{jk}^h = \mathring{C}_{jk}^h - \delta_j^h \lambda_k \tag{3.5}$$

este o conexiune finsleriană conform aproape simplectică.

c) Deoarece U_{pk}^q şi V_{pk}^q sînt arbitrari să considerăm

$$U_{pk}^q = -2\delta_k^q \omega_p + U'^q_{pk}, \quad V_{pk}^q = -2\delta_k^q \lambda_p + V'^q_{pk}$$

şi atunci formulele (3.1) devin:

$$\left\{\begin{aligned} N_j^h &= \mathring{N}_j^h + A_j^h, \\ F_{jk}^h &= \mathring{F}_{jk}^h - \delta_j^h \omega_k - \delta_k^h \omega_j + a_{jk} a^{ph} \omega_p - \frac{1}{2} A_k^q a^{ph} \frac{\partial a_{pj}}{\partial y^q} + \theta_{jq}^{ph} U'^q_{pk}, \\ C_{jk}^h &= \mathring{C}_{jk}^h - \delta_j^h \lambda_k - \delta_k^h \lambda_j + a_{jk} a^{ph} \lambda_p + \theta_{jq}^{ph} V'^q_{pk}. \end{aligned}\right. \tag{3.1'}$$

4. *Transformări de conexiuni finsleriene* conform aproape simplectice. Fie $(\underset{1}{N}{}_j^h, \underset{1}{F}{}_{jk}^h, \underset{1}{C}{}_{jk}^h)$ şi $(\underset{2}{N}{}_j^h, \underset{2}{F}{}_{jk}^h, \underset{2}{C}{}_{jk}^h)$ două conexiuni finsleriene conform aproape simplectice. Din (2.9) rezultă

$$\left\{\begin{aligned} \underset{a}{N}{}_j^h &= \mathring{N}_j^h + \underset{a}{A}{}_j^h, \quad (a = 1, 2); \\ \underset{a}{F}{}_{jk}^h &= \mathring{F}_{jk}^h + \frac{1}{2} a^{ph} \left(\mathring{\nabla}_k a_{pj} - 2\underset{a}{\omega}_k a_{pj} - \underset{a}{A}{}_k^q \frac{\partial a_{pj}}{\partial y^q} \right) + \underset{a}{P}{}_{jk}^h, \quad (a = 1, 2); \\ \underset{a}{C}{}_{jk}^h &= \mathring{C}_{jk}^h + \frac{1}{2} a^{ph} (\mathring{\nabla}_k a_{pj} - 2\underset{a}{\lambda}_k a_{pj}) + \underset{a}{Q}{}_{jk}^h, \quad (a = 1, 2). \end{aligned}\right. \tag{4.1}$$

Legătura dintre $(\underset{1}{N}{}^h_j, \underset{1}{F}{}^h_{jk}, \underset{1}{C}{}^h_{jk})$ şi $(\underset{2}{N}{}^h_j, \underset{2}{F}{}^h_{jk}, \underset{2}{C}{}^h_{jk})$ se obţine din relaţiile (4.1) prin scădere. Dacă în relaţiile obţinute notăm $(\underset{2}{N}{}^h_j, \underset{2}{F}{}^h_{jk}, \underset{2}{C}{}^h_{jk})$ prin $(\bar{N}^h_j, \bar{F}^h_{jk}, \bar{C}^h_{jk})$ şi $(\underset{1}{N}{}^h_j, \underset{1}{F}{}^h_{jk}, \underset{1}{C}{}^h_{jk})$ prin $(N^h_j, F^h_{jk}, C^h_{jk})$, iar diferenţele de tensori ce mai apar printr-un singur tensor, avem

$$(4.2)\qquad \begin{cases} \bar{N}^h_j = N^h_j + A^h_j, \\ \bar{F}^h_{jk} = F^h_{jk} - \dfrac{1}{2} a^{ph} \dfrac{\partial a_{pj}}{\partial y^q} A^q_k - \omega_k \delta^h_j + P^h_{jk}, \\ \bar{C}^h_{jk} = C^h_{jk} - \lambda_k \delta^h_j + Q^h_{jk}. \end{cases}$$

Aceste formule dau transformările de conexiuni finsleriene conform aproape simplectice.

Cu ajutorul formulelor de transformare (4.2) se poate arăta uşor că mulţimea transformărilor de conexiuni finsleriene conform aproape simplectice formează grup abelian în raport cu compunerea obişnuită a aplicaţiilor : $(N, F, C) \to (\bar{N}, \bar{F}, \bar{C}) \to (\bar{\bar{N}}, \bar{\bar{F}}, \bar{\bar{C}})$.

5. *Condiţii de integrabilitate* ale ecuaţiilor

$$(2.1)\qquad a_{ij|k} = 2\omega_k a_{ij}, \qquad a_{ij}/_k = 2\lambda_k a_{ij}.$$

Să considerăm formulele care dau legea de comutare pentru derivatele covariante ale tensorului

$$(5.1)\qquad \begin{cases} a_{ij|k|l} - a_{ij|l|k} = -a_{hj} R^h_{i\,kl} - a_{ih} R^h_{j\,kl} - a_{ij}|_h R^h_{kl} - a_{ij|h} T^h_{kl}, \\ a_{ij|k}|_l - a_{ij}|_{l|k} = -a_{hj} P^h_{i\,kl} - a_{ih} P^h_{j\,kl} - a_{ij|h} C^h_{kl} - a_{ij}|_h P^h_{kl}, \\ a_{ij}|_k|_l - a_{ij}|_l|_k = -a_{hj} S^h_{i\,kl} - a_{ih} S^h_{j\,kl} - a_{ij}/_h S^h_{kl}. \end{cases}$$

Dacă ţinem seama în paranteza (5.1) de (2.1) şi relaţiile obţinute se înmulţesc cu a^{jp}, după contracţie şi punerea în evidenţă a operatorului θ^{*pj}_{ih} găsim

$$(5.2)\qquad \begin{cases} \theta^{*jp}_{ih} R^h_{j\,kl} = -\delta^p_i (\omega_{k|l} - \omega_{l|k} + \lambda_h R^h_{kl} + \omega_h T^h_{kl}), \\ \theta^{*jp}_{ih} P^h_{j\,kl} = -\delta^p_i (\omega_k|_l - \lambda_{l|k} + \omega_h C^h_{kl} + \lambda_h P^h_{kl}), \\ \theta^{*jp}_{ih} S^h_{j\,kl} = -\delta^p_i (\lambda_k|_l - \lambda_l|_k + \lambda_h S^h_{kl}). \end{cases}$$

Luînd $p = i$ în (5.2) rezultă

$$(5.3)\qquad \begin{cases} R^j_{j\,kl} \equiv X_{kl} = -n(\omega_{k|l} - \omega_{l|k} + \lambda_h R^h_{kl} + \omega_h T^h_{kl}), \\ P^j_{j\,kl} \equiv Y_{kl} = -n(\omega_k|_l - \lambda_{l|k} + \omega_h C^h_{kl} + \lambda_h P^h_{kl}), \\ S^j_{j\,kl} \equiv Z_{kl} = -n(\lambda_k|_l - \lambda_l|_k + \lambda_h S^h_{kl}). \end{cases}$$

Dacă notăm

$$(5.4)\qquad R^{*h}_{j\,kl} = R^h_{j\,kl} - \frac{1}{n}\delta^h_j X_{kl}, \quad P^{*h}_{j\,kl} = P^h_{j\,kl} - \frac{1}{n}\delta^h_j Y_{kl}, \quad S^{*h}_{j\,kl} = S^h_{j\,kl} - \frac{1}{n}\delta^h_j Z_{kl}$$

şi ţinem seama de (5.2) rezultă

$$(5.5)\qquad \theta^{*jp}_{ih} R^{*h}_{j\,kl} = 0, \qquad \theta^{*jp}_{ih} P^{*h}_{j\,kl} = 0, \qquad \theta^{*jp}_{ih} S^{*h}_{j\,kl} = 0.$$

Dacă luăm h-derivata covariantă în (5.5) putem scrie

$$(5.6)\quad \begin{cases} \theta^{*jp}_{ih} R^{*h}_{j\,kl\,|\,l_1} = 0, \\ \theta^{*jp}_{ih} P^{*h}_{j\,kl\,|\,l_1} = 0, \\ \theta^{*jp}_{ih} S^{*h}_{j\,kl\,|\,l_1} = 0\,; \end{cases} \qquad \begin{cases} \theta^{*jp}_{ih} R^{*h}_{j\,kl\,|\,l_1\,|\,\ldots\,|\,l_r} = 0, \\ \theta^{*jp}_{ih} P^{*h}_{j\,kl\,|\,l_1\,|\,\ldots\,|\,l_r} = 0, \\ \theta^{*jp}_{ih} S^{*h}_{j\,kl\,|\,l_1\,|\,\ldots\,|\,l_r} = 0. \end{cases}$$

În mod analog luînd v-derivata covariantă în (5.5) rezultă

$$(5.7)\quad \begin{cases} \theta^{*jp}_{ih} R^{*h}_{j\,kl}\big|_{l_1} = 0, \\ \theta^{*jp}_{ih} P^{*h}_{j\,kl}\big|_{l_1} = 0, \\ \theta^{*jp}_{ih} S^{*h}_{j\,kl}\big|_{l_1} = 0\,; \end{cases} \qquad \begin{cases} \theta^{*jp}_{ih} R^{*h}_{j\,kl}\big|_{l_1}\big|\ldots\big|_{l_r} = 0, \\ \theta^{*jp}_{ih} P^{*h}_{j\,kl}\big|_{l_1}\big|\ldots\big|_{l_r} = 0, \\ \theta^{*jp}_{ih} S^{*h}_{j\,kl}\big|_{l_1}\big|\ldots\big|_{l_r} = 0. \end{cases}$$

Am obţinut astfel

Teorema 5. *Tensorii de curbură* $R^h_{j\,kl}$, $P^h_{j\,kl}$, *şi* $S^h_{j\,kl}$ *ai unei conexiuni finsleriene compatibile cu o structură conform aproape simplectică satisfac relaţiile* (5.6) *şi* (5.7).

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Institutul politehnic, Iaşi,
Catedra de matematici

BIBLIOGRAFIE

1. Cartan E., *Les espaces de Finsler.* Actualités scientifiques et industrielles, *79*, Paris, 1934.
2. Matsumoto M., *The Theory of Finsler Connections.* Publ. of the Study Group of Geometry 5, College of Liberal Arts and Sci., Okayama Univ., Japan, 1970.
3. Matsumoto M., *A Finsler Connection with Many Torsions.* Tensor N.S., 17 (1966).
4. Miron R., *Asupra conexiunilor compatibile cu structuri conform aproape simplectice sau conform metrice.* An. Univ. Timişoara, ser. Şt. mat. fiz., V (1967).
5. Miron R., *Espace a structure conforme presque symplectique.* Coll. math., Warszawa, XXVI (1972).
6. Rund H., *The Differential Geometry of Finsler Spaces.* Springer - Verlag, Berlin, 1959.
7. Ghinea I., *Connexions finsleriennes conformes métriques.* An. Univ. Timişoara, ser. Şt. mat. fiz. (1976) (sub tipar).
8. Ghinea I., *Connexions finsleriennes presque symplectiques.* An. Univ. Timişoara, ser. Şt. mat. fiz. (1977) (sub tipar).

CONNEXIONS FINSLERIENNES CONFORMES PRESQUE SYMPLECTIQUES

(Résumé)

En considérant la définition de la connexion finslerienne dans le fibré finslerien d'après M. Matsumoto [2], [3] on établi la forme générale des connexions finsleriennes compatibles à une structure conforme presque symplectique et on trouve les formules qui donnent les transformations des connexions conformes presque symplectiques et les conditions de compatibilité d'une connexion finslerienne avec la structure conforme presque symplectique.

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ON A STOCHASTIC VOLTERRA EQUATION

BY

MARICA LEWIN

This Note aims at proving the existence and uniqueness of a solution for the equation

$$x(t,\omega) = h(t,\omega) + \int_0^t k_1(t,\tau,\omega) f(\tau, x(\tau,\omega))\, d\tau + \tag{1}$$
$$+ \int_0^t k_2(t,\tau,\omega) G(\tau, x(\tau,\omega))\, dW, \quad t \geqslant 0$$

where

a) $W(t,\omega)$ is a normalized Wiener process on the probability triple (Ω, U, p).

b) The free term $h(t,\omega)$ has to be a real function, measurable in (t,ω) with

$$\left(\int_\Omega |h(t,\omega)|^2\, dP(\omega)\right)^{1/2} = (E\,|h(t,\omega)|^2)^{1/2} < \infty, \quad t \geqslant 0;$$

$h(t,\omega)$ is a process independent of $B_+ = U(W_s - W_t,\ s \geqslant t)$ the σ algebra generated by the increments of the Wiener process $W_s - W_t$, $s \geqslant t$, for every $t \geqslant 0$.

c) The kernels $k_1(t,\tau,\omega)$ and $k_2(t,\tau,\omega)$ are real functions, measurable in (t,τ,ω) independent of B_+.

We assume the mapping $(t,\tau) - k_1(t,\tau,\omega)$ to be continuous from $\Delta = \{(t,\tau),\ 0 \leqslant \tau \leqslant t < \infty)$ in $L_\infty(\Omega, U, P)\}$.

d) Let F_t, $t \geqslant 0$ denote a family of σ algebras which are nonanticipating with respect to the Wiener process $W(t)$ and $F_t \subset U$.

By the σ-algebra F_t, for the equation (1) we shall consider the smallest σ-algebra with respect to which the functions $h(\tau,\omega)$, $k_1(t,\tau,\omega)$, $k_2(t,\tau,\omega)$, $W(\tau)$ are measurable for $0 \leqslant \tau \leqslant t$, i.e. F_t is the σ-algebra generated by $h(\tau,\omega)$, $k_1(t,\tau,\omega)$, $k_2(t,\tau,\omega)$, $W(\tau)$, $0 \leqslant \tau \leqslant t$.

e) $G(t, x)$, $f(t, x)$ are continuous for (t, x), $t \geqslant 0$, $x \in R$. These conditions are so that the two integrals of the equation be meaningful.

D e f i n i t i o n 1. We define $C_2 = C(R_+, L_2(\Omega, U, P))$ as the space of all measurable functions, nonanticipating with respect to the family F_t for every $t \geqslant 0$, continuous from R_+ into $L_2(\Omega, U, P)$ with the topology of uniform convergence on every interval $[0, t]$.

Similarly we define $C_4 = C(R_+, L_4(\Omega, U, P))$.

The spaces C_2, C_4 are local convex spaces, whose topology is defined by means of the family of seminorms :

$$\sup_{0 \leqslant t \leqslant n} \left\{ \int_\Omega |x(t, \omega)|^2 \, dP(\omega) \right\}^{1/2} = \sup_{0 \leqslant t \leqslant n} \{Ex^2\}^{1/2} = \sup_{0 \leqslant t \leqslant n} \| x \|_{L_2(\Omega, U, P)}$$

in C_2 and

$$\sup_{0 \leqslant t \leqslant n} \left[\int_\Omega |x(t, \omega)|^4 \, dP(\omega) \right]^{1/4} = \sup_{0 \leqslant t \leqslant n} (Ex^4)^{1/4} = \sup_{0 \leqslant t \leqslant n} \| x \|_{L_4(\Omega, U, P)} .$$

D e f i n i t i o n 2. The pair of spaces (B, D) will be called admissible with respect to the operator T, $T : E \to F$, if $B \subset E$, $D \subset F$, and $TB \subset D$.

L e m m a 1. *Let T be a continuous operator from C_4 into C_2. If B, D are two Banach spaces, $B \subset C_4$, $D \subset C_2$, and the pair (B, D) is admissible with respect to the operator T, then T is a continuous operator from B into D. The result remains valid for $T : C_2 \to C_2$, and $B, D \subset C_2$.*

This result follows easily from the closed graph theorem.

D e f i n i t i o n 3. $x(t, \omega)$ is called a solution of (1) if :

a) $x(t, \omega)$ is F_t measurable, for every $t \geqslant 0$;

b) The integrals of the equation (1) are defined ;

c) The equation (1) is satisfied for every $t \in R_+$, with probability 1.

We shall look for a solution in the space C_2.

Let the operator $T_2 x = \int_0^t k_2(t, \tau, \omega) \, x(\tau, \omega) \, dW$, $k_2(t, \tau, \omega)$ is a measurable function in (t, τ, ω) so that

$$\sup_{(t, \tau)} \left(\int_\Omega k_2(t, \tau, \omega)^4 \, dP(\omega) \right)^{1/4} < \infty .$$

L e m m a 2. *Under the previous conditions, the operator*

$$T_1 : x \to \int_0^t k_1(t, \tau, \omega) \, x(\tau, \omega) \, d\tau$$

is a linear continuous operator from the space C_2 into itself and the operator

$$T_2 : x \to \int_0^t k_1(t, \tau, \omega) \, x(\tau, \omega) \, d\tau$$

is a linear, continuous operator from C_4 into C_2.

We shall prove for the operator T_2 (for T_1 is proved in [3]).

For proving the continuity of T_2, we must show that $\sup\limits_{t \leqslant n} (Ex^4(t, \omega))^{1/4}$, $x(t, \omega) F_t$ measurable implies

$$\sup_{t \leqslant n} \left(E \left(\int_0^t k_2(t, \tau, \omega)\, x(\tau, \omega)\, \mathrm{d}W \right)^2 \right)^{1/2} < \infty \tag{2}$$

so, applying the a) property of the Wiener integral [5]

$$E \left(\int_0^t k_2(t, \tau, \omega)\, x(\tau, \omega)\, \mathrm{d}W \right)^2 = \int_0^t E(k_2(t, \tau, \omega))\, x(\tau, \omega)^2\, \mathrm{d}\tau, \tag{3}$$

which, by Schwarz-Buniakovsky inequality, becomes

$$\int_0^t (E\, k_2^4(t, \tau, \omega))^{1/2} (E(x(\tau, \omega))^4)^{1/2}\, \mathrm{d}\tau \leqslant$$

$$\leqslant (\sup_{(\tau > 0, t > 0)} \| k_2(t, \tau, \omega) \|)^2_{L_4(\Omega, U, P)} (\sup_{t \leqslant n} \| x(t, \omega) \|)^2_{L_4(\Omega, U, P)}\, t, \quad \text{for } t \leqslant n;$$

(3) implies (2).

The linearity of T_2 is obvious, knowing the properties of the Wiener integral.

Theorem 1. *Given equation* (1) *where* h, k_1, k_2 *are the functions considered above, and the following conditions are fulfilled:*

1) *B and D are Banach spaces $B, D \subset C_2$ stronger than $C_2 = C(R_+ L_2(\Omega, U, P))$ so that, they are admissible with respect to the operator T_1 defined above.*

2) *There is a Banach spaces $B, D \subset C_2$ stronger than C_4, so that the pair (B_1, D) be admissible in point of the operator T_2,*

3) *$x \to f(t, x)$ is an operator on*

$$S = \{x(t, \omega),\ x(t, \omega) \in D,\ \| x(t, \omega) \|_D \leqslant \rho\}$$

with values in the Banach space B, which satisfies $\| f(t, x) - f(t, y) \|_B \leqslant \lambda_1 \| x - y \|_D$; for $x(t, \omega), y(t, \omega) \in S$ and λ_1 constant; $x \to G(t, x)$ is an operator defined on S with values in the Banach space B_1, and

$$\| G(t, x) - G(t, y) \|_{B_1} \leqslant \lambda_2 \| x(t, \omega) - y(t, \omega) \|_D;$$

4) $$h(t, \omega) \in D.$$

Then there exists a unique solution of the equation (1), *provided that*

a) $\lambda_1 k_1 + \lambda_2 k_2 \leqslant 1$, *where* $k_1 = \| T_1 \|$, $k_2 = \| T_2 \|$

and

b) $\| h(t, \omega) \|_D + \| f(t, 0) \|_B + \| G(t, 0) \|_{B_1} \leqslant (1 - \lambda_1 k_1 - \lambda_2 k_2) \rho.$

P r o o f. Let V be the operator

$$V(x) = h(t, \omega) + \int_0^t k_1(t, \tau, \omega) f(\tau, x)\, d\tau + \int_0^t k_2(t, \tau, \omega) G(\tau, x)\, dW,$$

operator defined on $S \subset D$, with values in D, V is a contraction operator from S into S, so that we can apply the Banach fixed point theorem and the equation (1) has a unique solution.

In fact $V_x - V_y \in D$

$$\| V_x - V_y \|_D \leqslant \left\| \int_0^t k_1 (f(\tau, x) - f(\tau, y))\, d\tau \right\|_D + \tag{4}$$

$$+ \left\| \int_0^t k_2 (G(\tau, x) - G(\tau, y))\, dW \right\|_D .$$

Because of the linearity and continuity, T_1 and T_2 are bounded and (4) becomes

$$\| V_x - V_y \|_D \leqslant k_1 \| f(t, x) - f(t, y) \|_B + k_2 \| G(t, x) - G(t, y) \|_{B_1} \leqslant$$

$$\leqslant (\lambda_1 k_1 + \lambda_2 k_2) \| x - y \|_D.$$

Now, let us prove $VS \subset S$

$$\| V(x) \|_D \leqslant \| h(t, \omega) \|_D + k_1 \| f(t_1, x(t, \omega)) \|_B + k_2 \| G(t, x) \|_{B_1}.$$

Because $\| f(t, x) \|_B \leqslant \| f(t, x) - f(t, 0) \| + \| \rho(t, 0) \|_B$ and similarly for $G(t, x)$ and using the Lipschitz conditions and the conditions (a), (b), we obtain :

1) $\| V(x(t, \omega)) \|_D \leqslant \rho$, V_x belongs to S.

Because V is a contraction operator and $VS \subset S$, one can apply the Banach fixed theorem, which implies the existence and uniqueness of a function $x(t, \omega) \in S$ so that $V_x(t, \omega) = x(t, \omega)$; $x(t, \omega)$ found is a solution for the equation (1) i.e. fulfils the conditions of the definition 1.

Remark. If $k_2(t, \tau, \omega)$ does not depend on ω, the theorem remains true and the operator $T_2 = \int_0^t k_2(t, \tau)\, x\, dW$ is a linear continuous operator from C_2 into C_2.

The solutions of the equation (1) can be throught as a stochastic process on $[0, \infty)$ and it can be regarded as a set of compatible, finite dimensional distributions

$$P(X_{t_1} \in B_1, \ldots, X_{t_n} \in B_n) = P_{t_1, t_2, \ldots, t_n}(B_1, B_2, \ldots, B_n).$$

The solution of the equation (1) is a Markov process.

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Polytechnic Institute, Jassy,
Department of Mathematics

REFERENCES

1. Corduneanu C., *Problèmes globaux dans la théorie des équations intégrales de Volterra.* Ann. Mat. Pura Appl., **67**, 349–363 (1965).
2. Corduneanu C., *Stability of some Linear Time-Varying Systems.* Mathematical Systems Theory, **3**, *2*, 151 – 156 (June 1969).
3. Tsokos C. P., *On a Stochastic Integral Equation of the Volterra Type.* Mathematical Systems Theory, **3**, *3*, 222 – 232 (September 1963).
4. Gheorghiu N., *Introducere în analiza funcţională.* Ed. Acad. R. S. R., Buc., 1974.
5. Гихман и Скороход, *Введение в теорию случайных процессов.* Изд. ,,Наука'', М., 1965.
6. Arnold L., *Stochastic Differential Equations. Theory and Applications.* J. Wiley, Interscience, 1974.

ASUPRA UNEI ECUAŢII STOCHASTICE DE TIP VOLTERRA

(Rezumat)

Obiectul acestei Note îl constutuie ecuaţia integrală stochastică de forma (1), unde $W(t)$ este un proces Wiener şi cea de-a doua integrală din membrul al doilea este o integrală Itô. Se studiază existenţa şi unicitatea soluţiei care se caută într-un anumit spaţiu de funcţii. Se foloseşte în acest scop metoda admisibilităţii introdusă pentru ecuaţii integrale deterministe de către C. Corduneanu.

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SENSITIVITY SYNTHESIS OF OPTIMAL STRATEGIES FOR DISTRIBUTED PARAMETER DIFFERENTIAL GAMES

I. SENSITIVITY FUNCTIONS, SENSITIVITY EQUATIONS, COMBINED SYSTEMS

BY

VICTOR FLORIN POTERAŞU

1. Introduction. As for distributed parameter differential games, there are works by R. J. Landy [2], L. C. Westphal, A. R. Stubberud [6], H. W. Kuhn, G. P. Szegö [5]. These works, however, were done for the systems with no system parameter variations and no uncertainty. Therefore it is necessary to consider another synthesis of optimal strategies with consideration of sensitivity to make up for the undesirable effects caused by system parameter variations and uncertainty. In this sensitivity synthesis, the central role is played by the combined system which consists of a model equation and its parameter sensitivity equation. By giving exanples, the effectiveness of the present sensitivity synthesis is illustrated. Another point of view, the differential game is regarded as the optimal control problem with an uncertainty g which is the opposite player to the control designer.

By considering this we can make up for the undesirable effects caused by system parameter variations and uncertainty in some sense large.

It is studied the case of two players zero sum distributed parameter differential games.

2. Sensitivity Functions and Sensitivity Equations. In this part, the sensitivity synthesis of optimal strategies is considered to make up for the state variations corresponding to various system parameter variations. This sensitivity synthesis was studied for optimal control problems in [7]. Only feedforward strategies are also here considered.

a) *α-sensitivity function and α-sensitivity equation.* We shall deal with the sensitivity for ordinary parameter variation without variation of initial conditions and the order of the system. Consider a system presented by

$$\frac{du}{dt} + A(t\,;\alpha)\,u = B_1(\alpha)f + B_2(\alpha)\,g, \quad u(0) = u_0\,, \tag{1}$$

where α is a variable parameter, $\alpha \in H_\alpha$; H_α being a Hilbert space. We shall regard the system (1) with parameter value α as the model system for a physical one. On the other hand we describe the actual system by (1) with parameter value $\alpha + \Delta\alpha$, $\Delta\alpha \in H_\alpha$. The difference in the state between the actual system and the model system corresponding to certain controls is described by

$$\Delta u(t\,;\alpha) = u(t\,;\alpha + \Delta\alpha) - u(t\,;\alpha) = v(t)\,\Delta\alpha + o\,[\,]\,\|\,\Delta\alpha\,\|^2_{H_\alpha}],$$

where $v(t) = u_\alpha(t;\alpha)$ is the α-sensitivity function. If A, B_1 and B_2 are differentiable in the parameter α, the sensitivity function satisfies

$$\frac{dv}{dt} + A(t;\alpha)\,v + A(t;\alpha)\,u = B_{1\alpha}(\alpha)f + B_{2\alpha}(\alpha)\,g, \quad v(0)=0. \tag{2}$$

Equation (2) is the α-sensitivity equation.

b) *β-sensitivity function and β-sensitivity equation.* Consider the case there is a difference Δu_0 only with respect to the initial condition between the model system and the actual one.

Here $\Delta u_0 = \varepsilon h$, $h \in I$ and ε is a scalar in some sense small. That is, the model system is described by

$$\frac{du}{dt} + A(t)\,u = B_1 f + B_2\,g, \quad u(0) = u_0\,.$$

The actual system is denoted by

$$\frac{du}{dt} + A(t)\,u = B_1 f + B_2\,g, \quad u(0) = u_0 + \Delta u_0\,. \tag{3}$$

Then the difference in the state between both systems is denoted by $\Delta u\,(t;u_0) = w\,(t)\,\Delta u_0 + o\,[\|\,\Delta u_0\,\|_I^2]$, where $w\,(t) = u_{u_0}\,(t, u_0)$ is the β-sensitivity function. The β-sensitivity equation

$$\frac{dw}{dt} + A(t)\,w = 0, \quad w\,(0) = h \tag{4}$$

is obtained.

c) *γ-sensitivity function and γ-sensitivity equation.* Suppose that the actual system is described by a differential difference equation :

$$\frac{du}{dt} + A(t)\,u + u\,(t-\varepsilon) = B_1 f + B_2\,g\,, \tag{5}$$

$$u(t) = h(t), \quad t \in (-\varepsilon, 0), \quad u(0) = u_0\,, \tag{6}$$

where ε is a scalar time delay parameter which is positive and in some sense small. Consider the model system

$$\frac{du}{dt} + A(t)\,u + u = B_1 f + B_2\,g, \quad u\,(0) = u_0\,, \tag{7}$$

which is obtained by letting $\varepsilon = 0$ in (5) and (6). Then the difference in the state between both systems is denoted by $\Delta u\,(t;\varepsilon) = \varepsilon v\,(t) + o\,(\varepsilon^2)$, where $u\,(t) = u_\varepsilon\,(t;0)$ is the γ-sensitivity function. The γ-sensitivity equation

$$\frac{dv}{dt} + A(t)\,v + v + A(t)\,u + u = B_1 f + B_2\,g, \quad v\,(0)=0 \tag{8}$$

is obtained, which is not a differential-difference equation.

3. Combined Systems. Consider the model equation and the sensitivity equation simultaneously, which is simply called the combined systems ; then it may be recognized that there is a possibility to control not only the model state but also its sensitivity function by choosing proper controls.

That is, we can take the effect of the small system parameter variations in the model equations into consideration at the initial stage of the synthesis. This is the basis of the sensitivity synthesis of optimal control. We extend this sensitivity synthesis to the differential games with terminal constraints. In cooperative games, the meaning of this synthesis can be easily understood. As in our minimax game the terminal state is previously given by a rule of the game, we can consider that both players want to make the state sensitivity small cooperatively. This is because they do not want to change the system state which has been synthesized once.

Combining the model equation and the α-sensitivity equation, we can get the α-combined system :

$$(9)\qquad \begin{aligned} &\frac{du}{dt} + A(t)\,u = B_1 f + B_2 g, \\ &\frac{dv}{dt} + A(t)\,v + A_\alpha(t)\,u = B_{1\alpha} f + B_{2\alpha} g\,, \quad u(0) = u_0, \quad v(0) = 0, \end{aligned}$$

which governs the extend state (u, v).

Similarly we can get the β-combined system and the γ-combined one. We consider, for example, the α-combined system.

Let

$$X = \begin{pmatrix} u \\ v \end{pmatrix}, \quad V = \begin{pmatrix} A & 0 \\ A_\alpha & A \end{pmatrix}, \quad Y = \begin{pmatrix} B_1 f + B_2 g \\ B_{1\alpha} f + B_{2\alpha} g \end{pmatrix},$$

and then (9) reduces to

$$(10)\qquad \frac{dX}{dt} + VX = Y, \qquad X(0) = \begin{pmatrix} u_0 \\ 0 \end{pmatrix}.$$

If $X(0) = 0$, then

$$\int_0^T \left(\frac{dX}{dt}, X\right) dt = \int_0^T \left\{\left(\frac{du}{dt}, u\right) + \left(\frac{dv}{dt}, v\right)\right\} dt = \frac{1}{2}[|u(T)|^2 + |v(T)|^2] \geqslant 0.$$

Moreover if $A \in L(v, v')$, $A_\alpha \in L(v, v')$, $(Au, u) \geqslant \beta_1 \| u \|_v^2$ and $((A + A_\alpha)u, u) \geqslant \beta_2 \| u \|_v^2$, $\beta_1, \beta_2 > 0$, $\forall u \in V$,

$$(11)\qquad \int_0^T (VX, X)\, dt = \int_0^T \{((A + A_\alpha)\, u_T\, u) + (Av, v)\}\, dt \geqslant \beta_2 \int_0^T \|u\|_v^2\, dt + \beta_1 \int_0^T \| v \|_v^2\, dt \geqslant 0.$$

Hence the existence and uniqueness of a solution of (10) is assured. This is true for the β and the γ-combined systems.

The sensitivity synthesis of optimal strategies for the problem is developed to make up for the undesirable effects caused by the system parameter variations. In the following part of this work we will study the sensitivity games with numerical examples — the comparison of trajectories for the sensitivity synthesis and conventional synthesis.

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Polytechnic Institute, Jassy,
Department of Mechanics

REFERENCES

1. Chang T. M., Wen C. Y., *Sensitivity Analysis in Optimal Systems Based on the Maximum Principle.* Ind. Eng. Chemistry Fundamentals, **7**, *3*, 422 — 429 (1968).
2. Landy R. J., *Distributed Parameter Differential Grames : A Dynamic Programming Approach.* Int. J. Control, **14**, *3*, 429 — 441 (1971).
3. Ichikawa K., Kanai K., *Optimal Control Synthesis of Distributed Parameter Systems.* Bull. of the JSME, **13**, *66*, 1426 — 1439 (1970).
4. Wang P. K. C., *Theory of Stability and Control for Distributed Parameter Systems* (a bibliography). Int. J. Control, **7**, *2*, 101 — 116 (1968).
5. Kuhn H. W., Szegö G. P., *Differential Games and Related Topics.* North. Holland, 1971.
6. Westphal L. C., Stubberud A. R., *On the Computation of Solutions of Certain Dynamic Games.* Int. J. Control, **21**, *2*, 193 — 202 (1975).
7. Poteraşu V. F., *Sensitivity Synthesis of Optimal Comtrol for Distributed Parameter Systems* (I). Bul. Inst. polit. Iaşi (in print).

SINTEZA SENZITIVITĂŢII STRATEGIILOR OPTIMALE PENTRU JOCURI DIFERENŢIALE CU PARAMETRI DISTRIBUIŢI

I. Funcţii de senzitivitate, ecuaţii de senzitivitate, sisteme combinate

(Rezumat)

Este studiată sinteza senzitivităţii strategiilor optimale pentru jocuri diferenţiale cu parametri distribuiţi pentru doi jucători cu sumă nulă. Se consideră variaţiile parametrilor ecuaţiilor modelului, se definesc tipurile de senzitivitate, funcţiile şi ecuaţiile de senzitivitate. Pentru studiu se consideră sistemul alcătuit din ecuaţia modelului şi ecuaţia senzitivităţii parametrilor, sistem numit combinat.

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MATEMATICĂ
MECANICĂ TEORETICĂ, FIZICĂ

D. C. 534.01

HARMONIC SOLUTIONS OF DUFFING'S EQUATION FORCED BY PERIODIC IMPULSES

BY

NORMAN J. FINIZIO

1. Introduction. Response diagrams for the forced Duffing's equation

$$\ddot{x} + \alpha x + \beta x^3 = F(t), \tag{1.1}$$

where α and β are real constants with $\dot{x} \equiv dx/dt$, $\alpha > 0$ and $F(t) = G \cos \omega t$, G a positive constant, are well known [4]. These response diagrams, obtained via an iteration scheme aimed at constructing a harmonic solution of Eq. (1.1), relate the amplitude, A, of this harmonic solution to its frequency ω.

In this paper we consider the case $F(t) = \mu \delta_s(t)$, where $\delta_s(t)$ is a periodic superposition of impulses and μ, the „amplitude" of F, is a constant. Our main objective is to construct harmonic motions the response diagrams of which exhibit the same characteristics as those for the classical case $F(t) = G \cos \omega t$.

In case $F(t) \equiv 0$, the phase plane trajectories associated with Eq. (1.1) are also well known [4]. We use these trajectories as building blocks for the construction of harmonic solutions of Eq. (1.1) for our $F(t)$ since phase plane trajectories in this case can be obtained from those in the case $F(t) \equiv 0$ by suitable instantaneous displacements in the velocity direction at the "impulse points".

We include in our discussion motions wherein the "impulse points" are taken on the velocity axis of the phase plane. A curious phenomenon exhibited by these motions is that the property of the solution as being in phase with the forcing function or out of phase with it reverses if β changes sign. This characteristic is not present in the classical case and does not occur in the other motions constructed here.

2. Harmonic Solutions to the Forced Duffing's Equation. 2.1. *The classical case* $F(t) = G \cos \omega t$. The equation

$$\ddot{x} + \alpha x + \beta x^3 = F(t), \tag{2.1.1}$$

where α and β are real constants, $\alpha > 0$, and $\cdot \equiv d/dt$ is known as the forced Duffing's equation [4]. The case $\beta > 0$ ($\beta < 0$) is refered to as the case of hard (solt) springs. Closed form solutions of Eq. (2.1.1) are not known but in the case $F(t) \equiv 0$ a phase plane analysis demonstrates the existence of periodic solutions [4]. The period and amplitude of such a pe-

riodic solution can be calculated once the system parameters are known [4]. The case $F(t) = G \cos \omega t$, G a positive constant, is well known and is presented in detail in the book by Stoker [4]. In this case if one seeks a harmonic solution of Eq. (2.1.1) which has amplitude A and frequency ω, then an iteration scheme yields the relations for the response diagrams, i.e., a plot of A versus ω. The case $G = 0$ is refered to as the free vibration.

One notes from these response diagrams that it is convenient to assume the amplitude of the response as known and to compute the corresponding value(s) of the frequency. Furthermore, if the amplitude A is given then it is possible to obtain two different harmonic solutions one which is in phase with the external force (in case $\omega_{\text{force}} < \omega_{\text{free}}$) and one which is out of phase with the external force (in case $\omega_{\text{free}} < \omega_{\text{force}}$). It is this fact which motivates our subsequent constructions.

In case $F(t) \equiv 0$ and $\beta > 0$, the phase plane geometry for Eq. (2.1.1) is also well known [4]. The closed trajectories in the phase plane are represented by the expression

$$v^2 + \alpha x^2 + \frac{\beta}{2} x^4 = h,$$

where $v = \dot{x}$ and h, the curve parameter, is twice the total energy of the system. Thus each closed trajectory represents a periodic solution of Eq. (2.1.1) whose amplitude A_{h_i} ($\equiv x_{\max}$) is given by [4]

$$A_{h_i}^2 = \frac{1}{\beta}\left[-\alpha + (\alpha^2 + 2\beta h_i)^{\frac{1}{2}}\right], \tag{2.1.2}$$

and whose period T_{h_i} is given by [4]

$$T_{h_i} = 4\int_0^{A_{h_i}} \left[h_i - \alpha x^2 - \frac{\beta}{2} x^4 \right]^{-\frac{1}{2}} dx. \tag{2.1.3}$$

The circular frequency ω_{h_i} is defined to be the ratio $2\pi/T_{h_i}$. The subtitution $x = A_{h_i} \sin \theta$ transforms (2.1.3) to the following [4]

$$T_{h_i} = 4\sqrt{2} \int_0^{\frac{\pi}{2}} [2\alpha + \beta A_{h_i}^2 + \beta A_{h_i}^2 \sin^2 \theta]^{-\frac{1}{2}} d\theta. \tag{2.1.4}$$

The phase curves in the case $\beta < 0$ are somewhat more complicated [4]. In this case, though, if one restricts the total energy to be such that $h_i < \alpha^2/(-2\beta)$ the phase curves are closed and resemble those for the case $\beta > 0$. For this restriction on h formulas (2.1.2) — — (2.1.4) remain valid.

2.2. *The case* $F(t) = \mu\delta_s(t)$. We see easily that one can obtain phase curves for Eq. (2.1.1) with $F(t) = \mu\delta_s(t)$, from the phase plane trajectories in the case $F(t) \equiv 0$ since the presence of an impulsive forcing function in Eq. (2.1.1) causes, in the phase plane, an instantaneous action in the vertical direction only. Thus it is a simple matter to construct periodic solutions of Eq. (2.1.1) in case $F(t)$ is a periodic impulse, e.g., the periodic motion depicted in Fig. 1.

P_1 and P_2 represent the points (in time) at which the impulse is to be applied, that is, the "impulse points". At P_1 the application of the impulse has the effect of removing energy from the system (since $h_1 > h_2$) and at P_2 that of putting energy back into the system. Clearly this motion is periodic and its period could be computed once h_1, h_2, P_1, and P_2 (and of course known, fixed α and β) are known (it is to be realized that only

three of these parameters are independent since in order to guarantee periodicity the magnitude of the strength of the impulse is to be the same at P_1 as at P_2). Contrary to the classical phase plane motions, the motion in Fig. 1 does not possess a symmetric amplitude. Thus one would have to define what is meant by the amplitude of such a motion. Furthermore it is not immediately clear that any such definition would admit two motions with the same amplitude but different frequencies. (Indeed this cannot be accomplished with this motion).

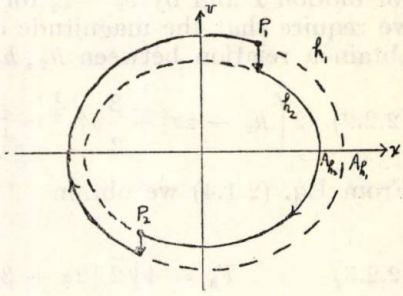

Fig. 1

Thus it is clear that it is easy to produse periodic solutions of Duffing's equation with periodic impulsive forcing function but the production of such motions which exhibit the characteristics of the classical case require some care.

We consider first the motions depicted in Fig. 2 and 3 which we refer to respectively as motion *I* and motion *II*.

Clearly both motions have the same (well defined) amplitude A_{h_2}; x_1 is to be chosen so that $0 < x_1 < A_{h_3}$.

One can imagine that both motions originate at the point P_4 and then depending on whether the "initial impulse" is positive or negative we obtain, respectively, motion *I* or motion *II*. Thus

$$T_{I(II)} = 2\tau_{h_{1(3)}} + T_{h_2} - 2\tau_{h_2},$$

where T_{h_2} is given by formula (2.1.3) with $i = 2$ and

$$\tau_{h_i} = \int_0^{x_1} \left[h_i - \alpha x^2 - \frac{\beta}{2} x^4 \right]^{-\frac{1}{2}} \mathrm{d}x =$$

(2.2.1)

$$= \sqrt{2} \int_0^{\sin^{-1}(x_1/A_{h_i})} [2\alpha + \beta A_{h_i}^2 + \beta A_{h_i}^2 \sin^2 \theta]^{-\frac{1}{2}} \mathrm{d}\theta.$$

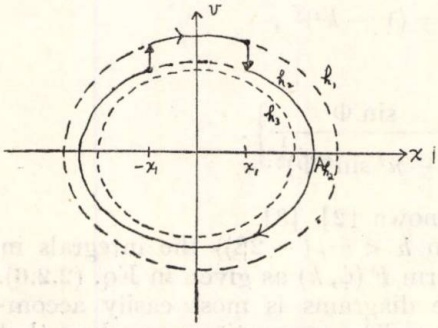

Fig. 2

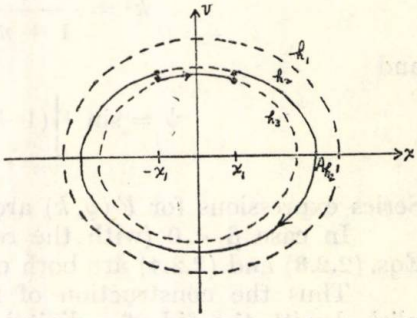

Fig. 3

Clearly $P_i \equiv \left(x_1, v_i = \left| h_i - \alpha x_1^2 - \frac{\beta}{2} x_1^4 \right|^{\frac{1}{2}}\right)$, $i = 1, 2, 3$. Using impulse momentum relations [1] the strength of the impulse is given by $v_2 - v_1$ for motion I and by $v_2 - v_3$ for motion II. In order to achieve periodicity we require that the magnitude of the strengths be the same and hence we obtain a relation between h_1, h_2, and h_3 which must hold, namely,

$$(2.2.2) \quad 2\left(h_2 - \alpha x_1^2 - \frac{\beta}{2} x_1^4\right)^{\frac{1}{2}} = \left(h_1 - \alpha x_1^2 - \frac{\beta}{2} x_1^4\right)^{\frac{1}{2}} + \left(h_3 - \alpha x_1^2 - \frac{\beta}{2} x_1^4\right)^{\frac{1}{2}}.$$

From Eq. (2.1.4) we obtain

$$(2.2.3) \qquad T_{h_i} = 4\sqrt{2}\,[2\alpha + \beta A_{h_i}^2]^{-\frac{1}{2}} \int_0^{\pi/2} [1 + n^2 \sin^2 \theta]^{-\frac{1}{2}}\, d\theta,$$

with $n^2 = \beta A_{h_i}^2/(2\alpha + \beta A_{h_i}^2)$ and from Eq. (2.2.1) we obtain

$$(2.2.4) \qquad \tau_{h_i} = \sqrt{2}\,(2\alpha + \beta A_{h_i}^2)^{-\frac{1}{2}} \int_0^{\sin^{-1}(x_1/A_{h_i})} [1 + n^2 \sin^2 \theta]^{-\frac{1}{2}}\, d\theta,$$

with n^2 as above. Thus the quantities T_{h_i} and τ_{h_i} may be calculated using well known formulas for elliptic integrals. The results differ depending on whether β is positive or negative.

In case $\beta > 0$ the integrals in Eqs. (2.2.3) and (2.2.4) are both of the form [2]

$$(2.2.5) \qquad \int_0^{\Phi} [1 + n^2 \sin^2 \theta]^{-\frac{1}{2}}\, d\theta = lF(\psi, k),$$

where

$$(2.2.6) \qquad F(\psi, k) = \int_0^{\psi} [1 - k^2 \sin^2 \theta]^{-\frac{1}{2}}\, d\theta, \quad k^2 < 1.$$

$$k^2 = \frac{n^2}{1 + n^2}, \quad l = (1 - k^2)^{\frac{1}{2}},$$

and

$$\psi = \sin^{-1}\left[(1 + n^2)^{\frac{1}{2}} \frac{\sin \Phi}{(1 + n^2 \sin^2 \Phi)^{\frac{1}{2}}}\right].$$

Series expressions for $F(\psi, k)$ are well known [2], [3].

In case $\beta < 0$ (with the restriction $h < \alpha^2/(-2\beta)$) the integrals in Eqs. (2.2.3) and (2.2.4) are both of the form $F(\psi, k)$ as given in Eq. (2.2.6).

Thus the construction of response diagrams is most easily accomplished with the aid of a digital computer. The computing procedure that

we followed was to prescribe α and β and consider them fixed. Then an initial value h_3 was prescribed and an acceptable x_1 computed, i.e., $x_1 = bA_{h_3}$, $0 < b < 1$. Considering x_1 now to be fixed we next prescribe an initial value of μ (assumed positive) and h_2, h_1 calculated on the basis of Eq. (2.2.2). With these values we then compute A_{h_2} (the amplitude), T_I, T_{II}, T_{h_2} (the free period), ω_I, ω_{II}, and ω_{h_2}. Next with μ fixed, h_3 is given a new initial value (greater than the previous h_3 so as to increase the amplitude) without changing x_1, and the above calculations made once again. The procedure is repeated to obtain a sufficient number of points to graph A_{h_2} versus ω_I, A_{h_2} versus ω_{II}, and A_{h_2} versus ω_{h_2} for this given value of μ. Next a new initial value for μ is taken and the process repeated (once again without changing x_1). Clearly one can take new initial values for μ as often as one pleases, each time two curves are obtained for the response diagram. Finally the entire process can be repeated (and a new response diagram obtained) with a different (acceptable) x_1. For both $\beta > 0$ and $\beta < 0$ if one fixes the amplitude and fixes μ one finds that $\omega_{II} < \omega_{h_2} < \omega_I$. Thus analogous to the classical case [4] one says that motion *II* is in phase with the external force and motion *I* is out of phase with the external force. That this should be the case appeals to our intuition since Fig. 2, for example, demonstrates that when the "amplitude" of the external force is negative the corresponding value of the response is positive and vice versa.

The response diagrams, Figs. 4 and 5 obviously are schematically the same as the classical case. These diagrams are drawn for $\alpha = 9$, $\beta = +1$ and $\alpha = 9$, $\beta = -1$. It is important to note that classical response diagrams are an approximation to the actual response diagrams [4] in that they derive from the first few terms in the iteration process, whereas Figs. 4 and 5 are accurate within the evaluation of the elliptic integrals.

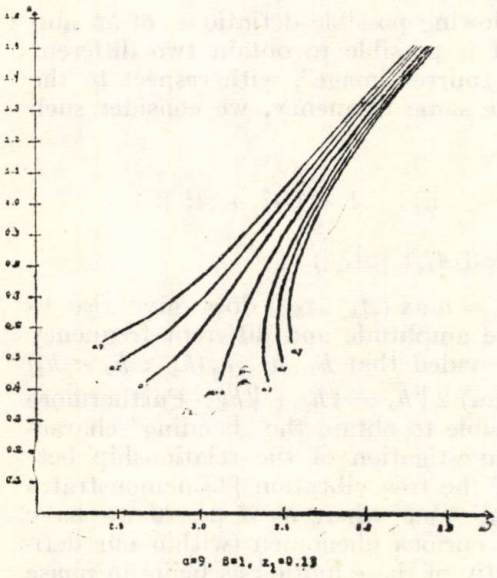

Fig. 4

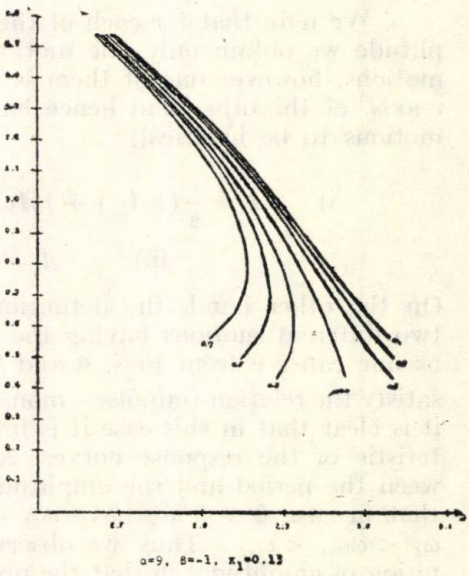

Fig. 5

A natural motion to consider would be that for which the "impulse points" are on the v-axis, but, as in Fig. 1, difficulties (with respect to obtaining the characteristics of the classical response diagrams) arise concerning the proper definition of an amplitude. To begin with if one tries to consider $x_1 \to 0$ in Figs. 2 and 3, the corresponding motions are not well defined (the limiting case that is) unless we require that at the "impulse points" the same amount of energy be alternately taken out of and put into the system. Even with this requirement (regardless of the definition of the amplitude) we obtain two motions with different amplitude. We conclude then that if motions corresponding to "impulse points" on the v-axis exist, they can not be obtained from the motions of Figs. 2 and 3 by taking $x_1 \to 0$.

With the "impulse points" on the v-axis one possibility is depicted in Figs. 6 and 7 (refered to respectively as motion I and moiton II).

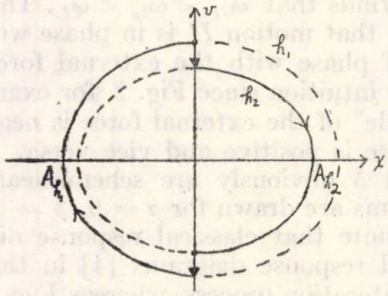

Fig. 6

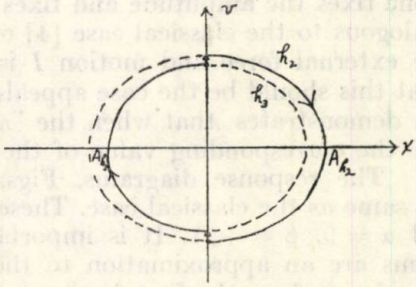

Fig. 7

We note that for each of the following possible defintions of an amplitude we obtain only one motion (it is possible to obtain two different motions, however one of them is the „mirror image", with respect to the v-axis, of the other and hence has the same frequency, we consider such motions to be identical):

$$\text{i)} \quad A = \frac{1}{2}(|A_{h_1}| + |A_{h_2}|); \qquad \text{ii)} \quad A = (A_{h_1}^2 + A_{h_2}^2)^{\frac{1}{2}};$$

$$\text{iii)} \quad A = \max(|A_{h_1}|, |A_{h_2}|).$$

On the other hand, the definition $A = \max(A_{h_1}\ A_{h_2})$ does give rise to two different motions having the same amplitude and different frequency as one can see from Figs. 6 and 7 provided that $h_1, h_3, h_3\ (h_3 < h_2 < h_1)$ satisfy the relation (impulse—momentum) $2\sqrt{h_2} = \sqrt{h_3} + \sqrt{h_1}$. Furthermore it is clear that in this case it is impossible to obtain the „bending" characteristic of the response curves. An investigation of the relationship between the period and the amplitude of the free vibration [4] demonstrates that in case $\beta > 0$ we have $\omega_{II} < \omega_{\text{free}} < \omega_I$ where as if $\beta < 0$ we have $\omega_I < \omega_{\text{free}} < \omega_{II}$. Thus we observe a curious phenomen (within our definition of amplitude) in that the propertiy of these motion as being in phase or out of phase with the external forcing function depends on the sign of β.

(Actually the same is true for the motions of Figs. 6 and 7 regardless of the definition of the amplitude). We remark that this phenomenon does not occur in the classical case nor did it occur in the motions of Figs. 2 and 3 where the order of magnitude of the frequency is independent of the type of spring considered (i.e., the sign of β).

Another type of motion (not obtainable from Figs. 2 and 3) whose response diagrams are schematically the same as the classical case Figs. 1 and 2 is given in Figs. 8 and 9 (refered to respectively as motion *I* and motion *II*).

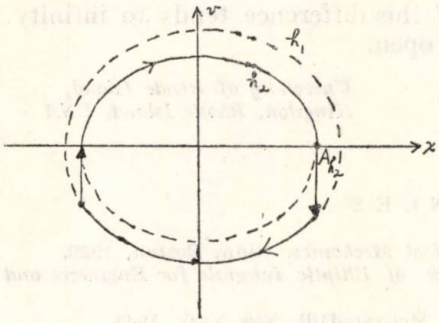

Fig. 8

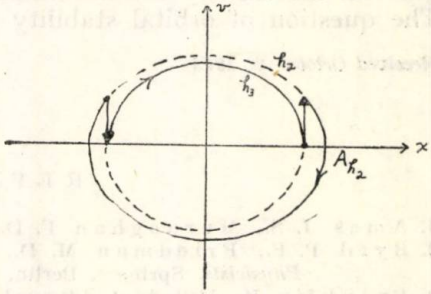

Fig. 9

It is clear that both of these motions have the same (well defined) amplitude A_{h_2}. In order that the magnitude of the strength of the impulse be the same in both cases, h_1, h_2, and h_3 ($h_3 < h_2 < h_1$) must satisfy the relation

$$h_3 = \alpha y^2 + \frac{\beta}{2} y^4,$$

where

$$y^2 = \frac{1}{\beta}\left[-\alpha + (\alpha^2 + 2\beta(2h_2 - h_1))^{-\frac{1}{2}}\right]. \tag{2.2.7}$$

In case $\beta > 0$ we must further restrict $h_1 < 2h_2$. Thus we have

$$T_I = \frac{1}{2} T_{h_2} + 2\tau_{h_1}, \quad T_{II} = \frac{1}{2} T_{h_3} + T_{h_2} - 2\tau_{h_2}$$

and

$$T_{\text{free}} = T_{h_2}.$$

T_{h_i} is given by Eq. (2.1.4), τ_{h_1} is given by Eq. (2.2.1) with x_1 replaced by A_{h_2} and τ_{h_2} is given by Eq. (2.2.1) with x_1 replaced by y (the positive square root of Eq. (2.2.7)). The evaluation of T_{h_i} and τ_{h_i} is accomplished once again through the use of Formulas (2.2.5) — (2.2.6) A computing scheme analogous to that described for the motions of Figs. 2 and 3 (without any reference to x_1) can be set up and response diagrams obtained.

Clearly there are many other different types of periodic motions that one can construct, e.g., the motion of Fig. 1. However, with the exception of those we have discussed (or "mirror" images of those discussed), one encounters the difficulty of suitable defining an amplitude and also the difficulty of producing two motions of different frequency but with the same amplitude exhibiting the characteristics of the classical case.

Finally we remark that each of the periodic motions of Figs. 2, 3, 6, 7, 8, and 9 are not asymptotically stable since a small perturbation in the initial point, e.g., P_4 in Figs. 2 and 3, produces a motion whose period differs from that of the unperturbed motion by a finite amount. Thus as time goes on indefinitely the accumulation of this difference tends to infinity. The question of orbital stability is still open.

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University of Rhode Island,
Kingston, Rhode Island, USA

REFERENCES

1. Ames J. S., Murnaghan F. D., *Theoretical Mechanics*. Ginn, Boston, 1929.
2. Byrd P. F., Friedman M. D., *Handbook of Elliptic Integrals for Engineers and Physicists*. Springer, Berlin, 1954.
3. Franklin P., *Methods of Advanced Calculus*. McGraw-Hill, New York, 1944.
4. Stoker J. J., *Nonlinear Vibrations in Mechanical and Electrical Systems*. Interscience, New York, 1950.

SOLUŢII ARMONICE ALE ECUAŢIEI DUFFING PRODUSE DE IMPULSURI PERIODICE

(Rezumat)

Se studiază soluţiile ecuaţiei (1) cînd $F(t) = \mu\delta_s(t)$, în care $\delta_s(t)$ este o suprapunere periodică de impulsuri şi μ = const. Pentru construcţia soluţiilor armonice se utilizează traiectoriile de fază bine cunoscute [4] din cazul $F(t) \equiv 0$, care sînt convenabil translate în direcţia vitezei.

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C.Z. 528.08

DETERMINAREA VECTORULUI DEPLASĂRII CONSTRUCŢIILOR PRIN MĂSURĂRI INDIRECTE PONDERATE

DE

GH. NISTOR şi ANETA BRAIER

1. Introducere. La determinarea simultană a vectorului deplasării orizontale a punctelor de control, de pe construcţia observată, în raport cu o serie de puncte fixe din reţeaua de microtriangulaţie, cea mai acceptabilă cale de urmat în operaţia de compensare a măsurărilor ar fi următoarea.

Cu coordonatele cunoscute ale punctelor din reţeaua de microtriangulaţie se compensează toate măsurările executate, separat, în toate ciclurile de observaţie, obţinînd de fiecare dată coordonatele punctelor de control. Din distribuţia coordonatelor acestor puncte, obţinute în diferitele cicluri de observaţii, se pot constata mărimile deplasărilor relative şi absolute.

Acest mod de rezolvare a problemei propuse, acceptat în mod firesc şi conducînd la rezultate din cele mai bune, este extrem de anevoios, necesitînd un volum mare de calcule.

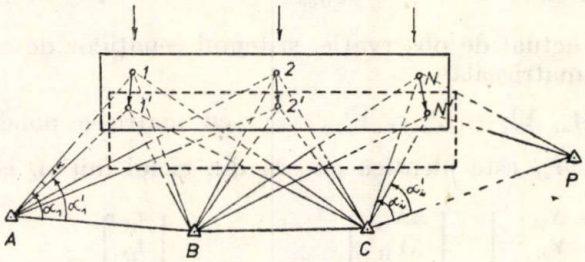

Fig. 1

Deoarece o serie de elemente ale operaţiei de tratare a măsurărilor efectuate sînt comune tuturor ciclurilor de observaţie, în cele ce urmează se va expune o metodă de determinare a vectorului deplasării în funcţie directă de elementele măsurate pe teren. Trebuie subliniat că, datorită condiţiilor dificile în care se realizează urmărirea comportării unor construcţii mari, supuse la diferite solicitări, din punctele fixe ale reţelei de microtriangulaţie, cea mai indicată metodă de tratare riguroasă a măsurărilor efectuate în diferite cicluri de observaţie, este metoda măsurărilor indirecte ponderate. Aceasta face obiectul cercetării de faţă.

2. Stabilirea algoritmului general de calcul. Se analizează determinarea componentelor deplasării orizontale a unui număr de N puncte de control, de pe construcţia observată, în raport cu P puncte fixe din reţeaua de microtriangulaţie, prin metoda intersecţiei multiple înainte, cazul măsurării unghiurilor.

În ciclul iniţial de observaţie, pentru determinarea poziţiei punctelor de control 1, 2,....N, prin intersecţie înainte, se scrie sistemul ecuaţiilor de erori, ce sînt exprimate prin ecuaţia matriceală

$$(1) \qquad A_{nk} X^0_{kl} + L^0_{nl} = V^0_{nl}, \qquad \text{cu matricea pondere } P^0_{nn},$$

unde s-a notat cu : n — numărul unghiurilor orizontale măsurate şi care este egal cu numărul ecuaţiilor de erori ; k — numărul corecţiilor coordonatelor, egal cu numărul necunoscutelor ($k = 2N$) ; A_{nk} — matricea coeficienţilor ecuaţiilor de erori, unde $n > k$; X^0_{kl} — vectorul corecţiilor ce se aplică coordonatelor aproximative pentru obţinerea coordonatelor compensate ; L^0_{nl} — vectorul termenilor liberi ai ecuaţiilor de erori ; V^0_{nl} — vectorul corecţiilor de aplicat unghiurilor măsurate pentru obţinerea mărimilor compensate ale acestora. Aceste matrice se pot scrie

$$A_{nk} = \begin{bmatrix} a_1 & b_1 \dots h_1 \\ a_2 & b_2 \dots h_2 \\ \dots & \dots \\ a_n & b_n \dots h_n \end{bmatrix}, \quad X^0_{kl} = \begin{bmatrix} X_{11} - X_{01} \\ Y_{11} - Y_{01} \\ \dots \\ X_{1N} - X_{0N} \\ Y_{1N} - Y_{0N} \end{bmatrix} = \begin{bmatrix} \Delta X_{01} \\ \Delta Y_{01} \\ \dots \\ \Delta X_{0N} \\ \Delta Y_{0N} \end{bmatrix},$$

$$L^0_{nl} = \begin{bmatrix} l_{01} \\ l_{02} \\ \dots \\ l_{0n} \end{bmatrix}, \quad V^0_{nl} = \begin{bmatrix} v_{01} \\ v_{02} \\ \dots \\ v_{0n} \end{bmatrix}, \quad P^0_{nn} = \begin{bmatrix} p_{01} & 0 \dots 0 \\ 0 & p_{02} \dots 0 \\ \dots & \dots \\ 0 & 0 \dots p_{0n} \end{bmatrix}.$$

În ciclul actual de observaţie, sistemul ecuaţiilor de erori este exprimat de ecuaţia matriceală

$$(2) \qquad A_{nk} X^1_{kl} + L^1_{nl} = V^1_{nl}, \qquad \text{cu matricea pondere } P^1_{nn},$$

unde matricea A_{nk} este identică cu cea din ciclul iniţial, iar celelalte sînt

$$X^1_{kl} = \begin{bmatrix} X_{21} - X_{01} \\ Y_{21} - Y_{01} \\ \dots \\ X_{2N} - X_{0N} \\ Y_{2N} - Y_{0N} \end{bmatrix} = \begin{bmatrix} \Delta X_{11} \\ \Delta Y_{11} \\ \dots \\ \Delta X_{1N} \\ \Delta Y_{1N} \end{bmatrix}, \quad L^1_{nl} = \begin{bmatrix} l_{11} \\ l_{12} \\ \dots \\ l_{1n} \end{bmatrix}, \quad V^1_{nl} = \begin{bmatrix} v_{11} \\ v_{12} \\ \dots \\ v_{1n} \end{bmatrix},$$

$$P^1_{nn} = \begin{bmatrix} p_{11} & 0 \dots\dots 0 \\ 0 & p_{12} \dots 0 \\ \dots & \dots 0 \\ 0 & \dots\dots p_{1n} \end{bmatrix}.$$

În ecuaţiile (1) şi (2) s-a notat cu : X_{0j} ($j = 1, 2,..., k$) coordonatele aproximative ale punctelor de control ; X_{1j} — coordonatele compensate

corespunzătoare ciclului iniţial; X_{2j} — coordonatele compensate corespunzătoare ciclului actual.

Vectorul deplasărilor orizontale ale punctelor de control, între cele două cicluri de observaţii, va fi dat de relaţia

$$X_{kl} = \begin{bmatrix} X_{21} - X_{11} \\ Y_{21} - Y_{11} \\ \cdots \\ X_{2N} - X_{1N} \\ Y_{2N} - Y_{1N} \end{bmatrix} = \begin{bmatrix} \Delta X_1 \\ \Delta Y_1 \\ \cdots \\ \Delta X_N \\ \Delta Y_N \end{bmatrix}. \tag{3}$$

Aşa cum s-a arătat în partea introductivă, acest mod de rezolvare presupune un volum mare de calcule, fiind complet neeconomic şi deci, nerecomandabil de aplicat în special cînd numărul punctelor de control, folosite la determinarea deplasării orizontale a construcţiei observate, este mare.

Întrucît doar mărimea elementelor măsurate pe teren diferă de la un ciclu la altul, celelalte fiind comune, se va căuta exprimarea unei relaţii directe între mărimile componentelor deplasării orizontale a punctelor de control şi variaţia în mărimea unghiurilor orizontale, măsurate în punctele fixe ale reţelei de microtriangulaţie. În acest scop se procedează astfel:

Se scade ecuaţia (1) din ecuaţia (2)

$$A_{nk}(X^1_{kl} - X^0_{kl}) + (L^1_{nl} - L^0_{nl}) = (V^1_{nl} - V^0_{nl}),$$

care se mai poate scrie

$$A_{nk} X_{kl} + L_{nl} = V_{nl}, \qquad \text{cu matricea pondere } P_{nn}, \tag{4}$$

unde matricea A_{nk} rămîne aceeaşi ca în ecuaţiile (1) şi (2) iar celelalte sînt

$$X_{kl} = \begin{bmatrix} (X_{21} - X_{01}) - (X_{11} - X_{01}) \\ (Y_{21} - Y_{01}) - (Y_{11} - Y_{01}) \\ \cdots \\ (X_{2N} - X_{0N}) - (X_{1N} - X_{0N}) \\ (Y_{2N} - Y_{0N}) - (Y_{1N} - Y_{0N}) \end{bmatrix} = \begin{bmatrix} \Delta X_{11} - \Delta X_{01} \\ \Delta Y_{11} - \Delta Y_{01} \\ \cdots \\ \Delta X_{1N} - \Delta X_{0N} \\ \Delta Y_{1N} - \Delta Y_{0N} \end{bmatrix} = \begin{bmatrix} \Delta X_1 \\ \Delta Y_1 \\ \cdots \\ \Delta X_N \\ \Delta Y_N \end{bmatrix},$$

$$L_{nl} = L^1_{nl} - L^0_{nl} = \begin{bmatrix} l_{11} - l_{01} \\ l_{12} - l_{02} \\ \cdots \\ l_{1n} - l_{0n} \end{bmatrix} = \begin{bmatrix} l_1 \\ l_2 \\ \cdots \\ l_n \end{bmatrix}, \quad V_{nl} = V^1_{nl} - V^0_{nl} = \begin{bmatrix} v_{11} - v_{01} \\ v_{12} - v_{02} \\ \cdots \\ v_{1n} - v_{0n} \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \cdots \\ v_n \end{bmatrix}.$$

În ecuaţia (4) matricea X_{kl} reprezintă vectorul componentelor deplasării orizontale a fiecărui punct de control, produsă între două cicluri de observaţii, L_{nl} — vectorul termenilor liberi, ale cărui elemente sînt formate de diferenţele dintre mărimile unghiurilor orizontale măsurate în ciclurile de observaţie actaul şi iniţial; V_{nl} — vectorul corecţiilor de aplicat diferenţelor provizorii dintre unghiurile orizontale pentru a se obţine valorile lor cele mai probabile.

Ecuaţia astfel obţinută poartă denumirea de ecuaţie de deformaţie sau de deplasare deoarece prin rezolvarea ei se obţine vectorul deplasării în funcţie directă de diferenţele unghiurilor orizontale măsurate. Conside-

rînd mărimile direct măsurate pe teren l_i ($i = 1, 2, \ldots, n$) ca independente şi că au erorile medii pătratice egale cu

$$m_i^2 = m_{0i}^2 + m_{1i}^2, \tag{5}$$

unde : m_{0i} — eroarea medie pătratică a unghiului orizontal, l_{0i}, măsurat în ciclul iniţial ; m_{1i} — eroarea medie pătratică a unghiului orizontal, l_{1i}, măsurat în ciclul actual, matricea pondere a ecuaţiei de deformaţii va fi

$$P_{nn} = \begin{bmatrix} \frac{\lambda^2}{m_{01}^2 + m_{11}^2} & 0 & \ldots & 0 \\ 0 & \frac{\lambda^2}{m_{02}^2 + m_{12}^2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \frac{\lambda^2}{m_{0n}^2 + m_{1n}^2} \end{bmatrix}. \tag{6}$$

Cînd precizia de măsurare a unghiurilor orizontale, în ambele cicluri de observaţie, este aceiaşi, erorile medii pătratice ale diferenţelor unghiulare, l_i, sînt egale cu

$$m_i^2 = m_{0i}^2 + m_{0i}^2 = 2m_{0i}^2 = 2m_{\alpha_i}^2, \tag{7}$$

iar matricea pondere capătă formă

$$P_{nn} = \begin{bmatrix} \frac{\lambda^1}{2m_{\alpha_1}^2} & 0 & \ldots & 0 \\ 0 & \frac{\lambda^2}{2m_{\alpha_2}^2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \frac{\lambda^2}{2m_{\alpha_n}^2} \end{bmatrix} = \begin{bmatrix} \frac{\lambda^2}{m_1^2} & 0 & \ldots & 0 \\ 0 & \frac{\lambda^2}{m_2^2} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \frac{\lambda^2}{m_n^2} \end{bmatrix}. \tag{8}$$

Determinarea vectorului deplasării punctelor de control, între cele două cicluri de observaţii, se face sub condiţia

$$V^T P V \to \min. \tag{9}$$

care, pentru cazul cercetat, se va exprima prin

$$V^T P V = X^T A^T P A X + 2X^T A^T P L + L^T P L \to \min. \tag{10}$$

Condiţia de minim va fi îndeplinită prin anularea derivatelor parţiale

$$\frac{\partial (V^T P V)}{2\partial (X)} = A^T P A X + A^T P L = 0,$$

sau

$$A_{kn}^T P_{nn} A_{nk} X_{nl} + A_{kn}^T P_{nn} L_{nl} = 0. \tag{11}$$

Ecuaţia matriceală obţinută corespunde sistemului de ecuaţii normale ale componentelor deplasării orizontale a punctelor de control.

Notînd cu $N_{kk} = A_{kn}^T P_{nn} A_{nk}$, ecuaţia (11) se poate scrie

$$N_{kk} X_{kl} + A_{kn}^T P_{nn} L_{nl} = 0. \tag{12}$$

Prin înmulţirea la stînga a ecuaţiei matriceale (12) cu inversa matricei N_{kk} se obţine vectorul deplasărilor (necunoscutelor)

$$X_{kl} = -N_{kk}^{-1} A_{kn}^{T} P_{nn} L_{nl} . \tag{13}$$

Făcînd notaţia

$$Q_{kk} = N_{kk}^{-1} = \begin{bmatrix} Q_{11} & Q_{12} \dots Q_{1k} \\ Q_{21} & Q_{22} \dots Q_{2k} \\ \dots & \dots \\ Q_{k1} & Q_{k2} \dots Q_{kk} \end{bmatrix}, \tag{14}$$

aceasta fiind matricea coeficienţilor de pondere, şi de asemenea

$$Z_{kn} = Q_{kk} A_{kn}^{T} P_{nn} , \tag{15}$$

se obţine pentru vectorul deplasării relaţia

$$X_{kl} = -Z_{kn} L_{nl} , \tag{16}$$

sau

$$\begin{bmatrix} \Delta X_1 \\ \Delta Y_1 \\ \dots \\ \Delta X_N \\ \Delta Y_N \end{bmatrix} = - \begin{bmatrix} Z_{11} & Z_{12} & \dots & Z_{1n} \\ Z_{21} & Z_{22} & \dots & Z_{2n} \\ Z_{k-1\,1} & Z_{k-1\,2} & \dots & Z_{k-1\,n} \\ \dots & \dots & \dots & \dots \\ Z_{k1} & Z_{k2} & \dots & Z_{kn} \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ l_{n-1} \\ \dots \\ l_n \end{bmatrix} .$$

Relaţia (16) îndeplineşte pe deplin condiţia impusă de exprimare a vectorului deplasării în funcţie directă de elementele măsurate pe teren.

La determinarea componentelor deplasării orizontale a punctelor de control, corespunzător mai multor cicluri de observaţii succesive, prin păstrarea preciziei de măsurare a unghiurilor orizontale în toate ciclurile de observaţii, matricea Z_{kn} va rămîne constantă. În acest fel, vectorul deplasării, corespunzător fiecărui ciclu de observaţii, se va obţine prin înmulţirea matricei Z_{kn}, de mărime constantă, cu vectorul termenilor liberi L_{nl}, ale cărui elemente se obţin direct din diferenţa unghiurilor orizontale măsurate în cele două cicluri de observaţii considerate.

Vectorul deplasării orizontale a fiecărui punct de control se calculează pe baza componentelor astfel

$$\alpha_{N1} = \alpha_{k/2,1} = \begin{bmatrix} L_1 \\ L_2 \\ \dots \\ L_N \end{bmatrix} = \begin{bmatrix} \sqrt{\Delta X_1^2 + \Delta Y_1^2} \\ \sqrt{\Delta X_2^2 + \Delta Y_2^2} \\ \dots \\ \sqrt{\Delta X_N^2 + \Delta Y_N^2} \end{bmatrix} . \tag{17}$$

În funcţie de aceleaşi elemente se calculează şi orientările vectorilor deplasării

$$\theta_{N1} = \theta_{k/2,1} = \begin{bmatrix} \theta_{L_1} \\ \theta_{L_2} \\ \\ \theta_{Ln} \end{bmatrix} = \begin{bmatrix} \operatorname{arc\,tg} \dfrac{\Delta Y_1}{\Delta X_1} \\ \operatorname{arc\,tg} \dfrac{\Delta Y_2}{\Delta X_2} \\ \cdot \quad \cdot \quad \cdot \\ \operatorname{arc\,tg} \dfrac{\Delta Y_N}{\Delta X_N} \end{bmatrix}. \tag{18}$$

În acest fel, problema propusă a fost rezolvată.

3. Concluzii. 1. Algoritmul propus permite obţinerea rapidă a vectorului deplasării orizontale a punctelor de control, de pe construcţia observată, funcţie directă de elementele măsurate pe teren. Prin tratarea prin metoda celor mai mici pătrate, mărimile obţinute sînt cele mai probabile.

2. Cu algoritmul prezentat, calculul se poate efectua în întregime la maşina electronică de calcul sau se poate calcula la maşina electronică doar matricea constantă, Z_{kn}, vectorul deplasării orizontale, X_{kl}, obţinîndu-se pentru fiecare ciclu de observaţie, direct pe teren, cu ajutorul unei maşini obişnuite de calcul.

3. Chiar în condiţiile în care precizia de măsurare a unghiurilor orizontale, în diferite cicluri de observaţii, diferă cu puţin, se poate păstra constantă matricea pondere, P_{nn}, şi deci, şi matricea Z_{kn}, avînd în vedere că variaţii ale ponderilor cu 10—15% determină modificări nesemnificative în rezultatele compensării.

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Institutul politehnic, Iaşi,
Laboratorul de topografie şi fotogrammetrie

ОПРЕДЕЛЕНИЕ ВЕКТОРА СМЕЩЕНИЯ СООРУЖЕНИЙ ПРИ ПОМОЩИ КОСВЕННЫХ ВЕСОВЫХ ИЗМЕРЕНИЙ

(Резюме)

Предлагается алгоритм для прямого вычисления вектора горизонтального смещения сооружений находящиеся пад горизонтальными напряжений, в зависимости от повторных измерениях злементов в поле, при помощи косвенных весовых измерений.

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ON THE THERMODYNAMICS OF SOME LINEAR FLOWS OF SIMPLE FLUIDS WITH MEMORY

BY

C. FETECĂU

1. Introduction. The aim of this work is to present some results in the thermodynamic of incompressible simple fluids with memory [1]. Such a class of linear flows, whose velocity field is given by (2.1) is investigated. It is find that the behaviour of a fluid in such flows is completely determined by five scalar material functionals.

In Sec. **3** the field equations for the determinance of the functions v and θ are specified. Finnaly, the steady case is considered and the field equations are completely solved.

2. Representantions of the Constitutive Functionals. The types of flows to be considered here are one-dimensional or lineal flows, for which the velocity field $\dot{x}(x, t)$ in a Cartesian co-ordinate system (x, y, z) has the form

$$\dot{x} = 0, \qquad \dot{y} = v(x, z, t), \qquad \dot{z} = 0. \tag{2.1}$$

Assuming that θ is a function of x, z and t only, we get the following constitutive eyuations (see [2])

$$\begin{aligned} &(\mathsf{T} = \mathsf{S} + p\mathsf{1}, \psi, \eta) = (\mathscr{H}, \mathop{\psi}_{s=0}^{\infty}, \eta)(\alpha^t(s), \beta^t(s), \theta^t(s)), \qquad \varepsilon = \psi + \theta\eta, \\ &\mathsf{q} = \mathop{\mathscr{F}}_{s=0}^{\infty}(\alpha^t(s), \beta^t(s), \theta^t(s)\,;\, g_1\, g_3), \qquad g_1 = \partial_x\theta, \qquad g_3 = \partial_z\theta, \end{aligned} \tag{2.2}$$

where S, T, q, ε, ψ and η are respectively, the stress tensor, the extra stress tensor, the heat flux, the internal energy, the free energy and the entropy at time t, p is indeterminate pressure, $\theta^t(.)$ is the temperature history and

$$\alpha^t(s) = \int_t^{t-s} \partial_x v(x, z, \sigma)\mathrm{d}\sigma, \qquad \beta^t(s) = \int_t^{t-s} \partial_z v(x, z, \sigma)\mathrm{d}\sigma. \tag{2.3}$$

It is well-known that the functionals from (2.2) are not arbitrary, they are restricted by the isotropy condition [1], [2]. Such, employing so-

me orthogonal transformations (see [2]) and following C a s w e l l [3] we conclude that

$$
(2.4)\quad
\begin{aligned}
&\mathcal{H}_{11} = \xi_1(\gamma^{t^2}, \mathcal{L}(\alpha^{t^2}), \theta^t), && \mathcal{H}_{22} = \xi_2(\gamma^{t^2}, \theta^t),\\
&\mathcal{H}_{33} = \xi_1(\gamma^{t^2}, \mathcal{L}(\beta^{t^2}), \theta^t), && \mathcal{H}_{12} = t(\gamma^{t^2}, l(\alpha^t), \theta^t),\\
&\mathcal{H}_{23} = t(\gamma^{t^2}, l(\beta^t), \theta^t), && \mathcal{H}_{13} = \xi_1(\gamma^{t^2}, \mathcal{L}(\alpha^t\beta^t), \theta^t),\\
&(\psi, \eta) = (\psi, \eta)(\gamma^{t^2}, \theta^t), && \gamma^{t^2} = \alpha^{t^2} + \beta^{t^2}, \qquad \alpha^{t^2} = (\alpha^t)^2,
\end{aligned}
$$

where the notations $l(\alpha)$ and $\mathcal{L}(\alpha\beta)$ are meant to indicate the linear and bilinear simetric dependence of α and β, respectively.

Remark 1. From $(2.4)_{1-6}$ we conclude that, the stress field in the linear flow (2.1) of a heat conducting incompressible simple fluid is determined by only three independent functionals[1]).

Now, returning to (2.2) and using some representation theorems from [5] we obtain in the same way

$$
(2.5)\quad
\begin{aligned}
&q_1 = k_1(\gamma^{t^2}, \omega^{t^2}, \mathcal{L}(\alpha^t\omega^t), \theta^t; g^2) + g_1 k_2(\gamma^{t^2}, \omega^{t^2}, \theta^t; g^2),\\
&q_2 = k_3(\gamma^{t^2}, \omega^{t^2}, l(\omega^t), \theta^t; g^2), \qquad q_3 = q_1(\beta^t, \alpha^t, \theta^t; g_3, g_1),
\end{aligned}
$$

where $\omega^t = \alpha^t g_1 + \beta^t g_2$ and $g^2 = g_1^2 + g_2^2$.

Remark 2. The heat flux in our linear flow (2.1) is completely determined by only two independent functionals.

So far, we have determined the nature of the response functionals, but for the proof of the existence of such motions it must be shown the compatibility of these functionals with the dynamical equations and the two laws of thermodynamics.

In the next section we reduce this problem to a system of two differential equations. In a well set problem, these equations and initial and boundary conditions are sufficient to determine v and θ.

3. Basic Balance Equations. In the usual assumption that the body force is $b = \operatorname{grad}\psi$ the equations of motion and energy balance become

$$
(3.1)\quad
\begin{aligned}
&\partial_x T_{11} + \partial_z T_{13} - \rho\partial_x\varphi = 0, \qquad \partial_x T_{13} + \partial_z T_{33} - \rho\partial_z\varphi = 0,\\
&\partial_x T_{12} + \partial_z T_{23} - \rho\partial_y\varphi = \rho\partial_t v,
\end{aligned}
$$

$$
(3.2)\quad S_{12}\partial_x v + S_{23}\partial_z v - \partial_x q_1 - \partial_z q_3 + \rho r = \rho\varepsilon_t,
$$

where $\varphi = \dfrac{P}{\rho} + \psi$ and ρ is the constant density. A simple analysis, using $(2.2)_1$, (2.3) and (3.1) lead us to

$$
(3.3)\quad
\begin{aligned}
&S_{11} = \partial_z h(x, z, t) + f(z, t) + a(t)\,y + \rho\psi,\\
&S_{22} = f_2(x, z, t) + a(t)\,y + \rho\psi,\\
&S_{33} = \int \partial_x^2 h(x, z, t)\,\mathrm{d}z + g(x, t) + a(t)\,y + \rho\psi,\\
&S_{13} = -\partial_x h(x, z, t),\\
&\partial_x S_{12} + \partial_z S_{23} + a(t) = \rho\partial_t v,
\end{aligned}
$$

---

[1]) The same result was obtained in [4, § 106] for locally viscometric flows ($\partial_z v = 0$) in the absence of the termic effects.

where a, f, g, h and f_2 are arbitrary functions of their arguments.

Along with (3.3) we have also (see [2])

$$S_{12} = 2\rho\delta_{\gamma^2}\psi(\gamma^{t^2}, \theta_d^t;\ \theta/\alpha^t), \qquad S_{23} = 2\rho\delta_{\gamma^2}\psi(\gamma^{t^2}, \theta_d^t;\ \theta/\beta^t),$$

$$\eta = -\partial_\theta\psi(\gamma^{t^2}, \theta_d^t;\ \theta) + \delta_\theta\psi(\gamma^{t^2}, \theta_d^t;\ \theta/1^t), \qquad 1^t(s) \equiv 1,$$

$$\gamma = \sigma - \frac{1}{\rho\theta}(q_1g_1 + q_2g_3) \geqslant 0, \tag{3.4}$$

$$\sigma = -\delta_\theta\psi(\gamma^{t^2}, \theta_d^t;\ \theta, \partial_x\theta^t) - 2\delta_{\gamma^2}\psi(\gamma^{t^2}, \theta_d^t;\ \theta/\alpha^t\partial_x v^t + \beta^t\partial_z v^t),$$

$$\theta_d^t(.) = \theta^t(.) - \theta 1^t(.),$$

where r is the heat supply per unit mass and $\delta_{\gamma^2}\psi$ and $\delta_\theta\psi$ are Fréchet derivatives of ψ.

Remark 3. The relations (3.4) are the restrictions imposed by the entropy inequality (in suitable smoothness assumptions upon the constitutive functionals). Together with (2.4) and (2.5) they are material restrictions and limit the classes of fluids which can flow in accordance with (2.1).

Remark 4. In view of (2.4), (2.5) and $(3.4_{1-3}$, it may said that the behaviour of our fluid in the flow (2.1) is determined by only five scalar material functionals.

Remark 5. For a given problem, (3.2) and $(3.3)_5$ form a complete system of functional differential equations for the functions v and θ (a is the driving force per unit volume in the y direction [2]).

4. Steady Linear Flows. In this case the functions v and θ does not depend of t and the relations (3.3) become

$$S_{11} = \partial_z h(x, z) + f(z) + ay + b(t) + \rho\psi, \qquad S_{13} = -\partial_x h(x, z),$$

$$S_{22} = f_2(x, z) + ay + b(t) + \rho\psi, \qquad S_{12} = \partial_z f_1(x, z),$$

$$S_{33} = \int \partial_x^2 h(x, z)\,dz + g(x) + ay + b(t) + \rho\psi, \tag{4.1}$$

$$S_{23} = -a - \partial_x f_1(x, z),$$

where f_1 is also an arbitrary function.

From (3.2) one obtains in the absence of the external radiation

$$q_1 = v(x, z)\,\partial_z f_1(x, z) - \partial_z \varkappa(x, z),$$

$$q_2 = -(a + \partial_x f_1(x, z))\,v(x, z) + \partial_x \varkappa(x, z), \tag{4.2}$$

with $\varkappa$ an arbitrary function of x and z. In (3.4) $\sigma \equiv 0$. The functions v and f must also satisfy ([4], § 108) $\partial_x v\partial_z f_1 - \partial_z v\partial_x f_1 - a\partial_z v \geqslant 0$.

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Polytechnic Institute, Jassy,
Department of Mathematics

REFERENCES

1. Coleman B. D., *Thermodynamics of Materials with Memory.* Arch. Rational Mech. Anal., 17, 1 – 46 (1964).
2. Fetecău C., *On the Thermodynamics of General Curvilinear Flows* (in print).
3. Caswell B., *Kinematics and Stress on a Surface of Rest.* Arch. Rational Mech. Anal., 26, 384 – 399 (1967).

4. Truesdell C., Noll W., *The Non-Linear Field Theories of Mechanics*. Handbuch der Physik, Vol. III/3, Springer, Berlin—Heidelberg—New York, 1965.
5. Wang C. C., *On Representations for Isotropic Functions* (I). Arch. Rational Mech. Anal., 33, 249 – 267 (1969).

ASUPRA TERMODINAMICII UNOR CURGERI LINIARE ALE FLUIDELOR SIMPLE CU MEMORIE

(Rezumat)

În această Notă sînt prezentate cîteva rezultate din termodinamica fluidelor simple în cazul unei curgeri liniare.

Utilizînd condiţia de izotropie se dă o reprezentare pentru funcţionalele de răspuns şi se stabilesc unele proprietăţi ale mişcării.

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EXTENSION FORMULAS FOR $SL(n, C)$: THE PRINCIPAL NONDEGENERATE SERIES

BY

GH. ZET and I. GOTTLIEB

1. Introduction. The unitary infinite-dimensional representations of certain noncompact Lie groups may be used in elementary particle physics. For such purposes, it is need to have the relevant representations in a basis made up of vectors belonging to finite-dimensional unitary irreducible representations of the maximal compact subgroup. In the cases of groups of the form $SU(p, q)$ and $SO(p,q)$ with p and q greater than unity, or $SL(n, R)$ and $SL(n, C)$ for $n \geq 3$, possible to be used as symmetry groups in physics, there is the multiplicity problem. Namely, in general, a unitary irreducible representation (UIR) of one of these groups contains a given finite-dimensional UIR of the maximal compact subgroup more than once. This problem is difficult and in fact there seems to be very few, if any, examples of noncompact groups whose UIR's involve multiplicity, and whose UIR'S have been completely worked out in a basis of UIR's of the maximal compact subgroup.

In some previous papers [1], [2] it has been outlined a method by which this problem can be solved. Namely, a noncompact Lie group G can be related, via group contraction, to another Lie group $S = K \times) T$ which is also noncompact but has a semidirect product structure. Here K denotes the maximal compact subgroup of G, and T an invariant subgroup of S. For suitably related UIR's of these two groups (G and S), the multiplicity structure of K representations is the same for G and S. To be specific, corresponding to UIR's of G, there are UIR's of S obtained from the former by group contraction, with property that the spectrum of K representations is the same in both cases and is not affected by contraction procedure.

In the case of the representations of S, a simple method [3] exist choosing labels to take care of the multiplicity problem. We can now take this advantage to solve the multiplicity problem for the group G. Namely, the generators of G can be wrtten as polynomial functions (the extension formulas) of the generators of suitably chosen unitary representations of S. Since the representations of S in a K basis are known [3], this provides a solution for UIR's of G.

In this work we construct the extension formulas, expressing the $SL(n, C)$ generators as functions of the $S = SU(n) \times) T_{n^2-1}$ generators, in the principal nondegenerate series. Here T_{n^2-1} denotes an $n^2 - 1$ - dimensional invariant subgroup of S. The maximally degenerate series of UIR of $SL(n, C)$ have been treated in a previous work [4], and the principal degenerate series will be taken elsewhere.

2. The Principal Nondegenerate Series of $SL(n, C)$. We consider the following two subgroups of $SL(n, C)$:

$$(2.1)\qquad Z \ni z = \begin{Vmatrix} 1 & 0 & 0 \dots 0 \\ z_1^2 & 1 & 0 \dots 0 \\ z_1^3 & z_2^3 & 1 \dots 0 \\ \cdot & \cdot & \cdot \\ z_1^n & z_2^n & z_3^n \dots 1 \end{Vmatrix}, \quad K \ni k = \begin{Vmatrix} k_1^1 & k_2^1 & k_3^1 \dots k_n^1 \\ 0 & k_2^2 & k_3^2 \dots k_n^2 \\ 0 & 0 & k_3^3 \dots k_n^3 \\ \cdot & \cdot & \cdot \\ 0 & 0 & 0 \dots k_n^n \end{Vmatrix},$$

where z_q^p $(p > q)$ are arbitrary complex numbers and $\det k = 1$. Then, almost every element $g \in SL(n, C)$ may be written in a unique way as the product of one element in K and one element in Z [5]:

$$(2.2)\qquad g = kz, \quad k \in Z, \quad z \in Z.$$

Namely, if we denote by $G^{p_1, p_2, p_3, \dots, p_m}_{q_1, q_2, q_3, \dots, q_m}$ the determinant of the minor composed from the matrix elements of g which are placed at intersection of the lines $p_1, p_2, p_3, \dots, p_m$ with the columnes $q_1, q_2, q_3, \dots, q_m$, then

$$(2.3)\qquad k_q^p = \frac{G^{p, q+1, \dots, n}_{p, q+1, \dots, n}}{G^{q+1, q+2, \dots, n}_{q+1, q+2, \dots, n}} \quad (p \leqslant q), \qquad z_q^p = \frac{G^{p, p+1, \dots, n}_{q, p+1, \dots, n}}{G^{p, p+1, \dots, n}_{p, p+1, \dots, n}} \quad (p > q).$$

In particular, the element $zg \in SL(n, C)$ may be written as

$$(2.4)\qquad zg = kz', \quad z' \in Z.$$

The representations $g \to R(g)$ of the principal nondegenerate series of $SL(n, C)$ are characterized by the integer numbers $m_2, m_3, \dots, m_n$ and the real numbers $\rho_2, \rho_3, \dots, \rho_n$. They are constructed in the Hilbert space $L^2(Z)$ of all functions $f(z)$ which are square-integrable; the action of the unitary operator $R(g)$ on a such function $f(z)$ is given by formula [5]:

$$(2.5)\qquad [R(g) f](z) = \alpha(k) f(z'); \qquad f(z), f(z') \in L^2(Z),$$

where

$$(2.6)\quad \alpha(k) = |k_2^2|^{m_2 + i\rho_2 - 2} (k_2^2)^{-m_2} |k_3^3|^{m_3 + i\rho_3 - 4} (k_3^3)^{-m_3} \dots |k_n^n|^{m_n + i\rho_n - 2(n-1)} (k_n^n)^{-m_n}.$$

Now, the formula (2.5) has the disadvantage that its right-hand side has a dependence on the continuous parameters $\rho_2, \rho_3, \dots, \rho_n$, when g is in $SU(n)$. Or it is known that the UIR's of $SU(n)$ are characterized by discrete indices only. This dependence may be removed if we consider an equivalent representation $R'(g) = SR(g) S^{-1}$, where S is a transformation in $L^2(Z)$, defined as:

$$(2.7)\qquad (Sf)(z) = \exp[i\,\Phi(z)] f(z),$$

where

$$(2.8)\quad \Phi(z) = \sum_{p=2}^{n} \left[\frac{1}{2} (\rho_{p-1} - \rho_p) \ln A_p \right], \quad A_p = \sum_{\substack{\nu_p, \dots, \nu_n = 1 \\ (\nu_p < \nu_{p+1} < \dots < \nu_n)}}^{n} |Z^{p, p+1, \dots, n}_{\nu_p, \nu_{p+1}, \dots, \nu_n}|^2.$$

Here, $Z^{p, p+1, \dots, n}_{\nu_p, \nu_{p+1}, \dots, \nu_n}$ is the determinant of the minor composed from the matrix elements of $z \in Z$ which are placed at intersection of the lines p,

$p+1,\ldots,n$ with the columnes ν_p, $\nu_{p+1},\ldots,\nu_n$. Then, the equivalent representation $R'(g)$ is given by

$$[R'(g)f](z) = \exp \mathrm{i}\,[\Phi(z) - \Phi(z')]\,f(z'). \tag{2.9}$$

We will now write the representation $R'(g)$ for the Lie algebra of $SL(n, C)$. This Lie algebra is $2(n^2-1)$-dimensional, being spanned by the generators J^α_β ($\alpha, \beta = 1, 2,\ldots, n$ and $J^\alpha_\alpha = 0$) of its maximal compact subgroup $SU(n)$ and by "noncompact" generators K^α_β ($\alpha, \beta = 1, 2,\ldots, n$ and $K^\alpha_\alpha = 0$). Everywhere in this work the sum convention over repeated indices is made. The structure equations of the Lie algebra of $SL(n, C)$ are :

$$\left\{\begin{aligned} [J^\alpha_\beta, J^\lambda_\mu] &= \delta^\alpha_\mu J^\lambda_\beta - \delta^\lambda_\beta J^\alpha_\mu, \\ [J^\alpha_\beta, K^\lambda_\mu] &= \delta^\alpha_\mu K^\lambda_\beta - \delta^\lambda_\beta K^\alpha_\mu, \\ [K^\alpha_\beta, K^\lambda_\mu] &= -\delta^\alpha_\mu J^\lambda_\beta + \delta^\lambda_\beta J^\alpha_\mu, \end{aligned}\right. \tag{2.10}$$

where δ^α_β is the Kronecker symbol. In addition, the following Hermiticity properties are satisfied

$$(J^\alpha_\beta)^+ = J^\beta_\alpha, \qquad (K^\alpha_\beta)^+ = K^\beta_\alpha. \tag{2.11}$$

If we consider g in (2.9) to be an element close to the identity the following expressions of the infinitesimal generators J^α_β and K^α_β are obtained :

$$\begin{aligned} J^\alpha_\beta = \sum_{p=2}^{n} \frac{m_p}{n}\,\delta^\alpha_\beta - \frac{m_p+2(p-1)}{2}\,(T^{\alpha p}_{p\beta} - T^{\alpha\, p+1}_{p+1\,\beta}) - \\ - \frac{m_p-2(p-1)}{2}\,(T^{\beta\, p*}_{\alpha\, p} - T^{\beta\, p+1*}_{p+1\,\alpha})\Big] + \sum_{\substack{\rho,\sigma=1\\(\rho<\sigma)}}^{n} (T^{\alpha\sigma}_{\rho\beta} - z^\sigma_\rho\, T^{\alpha\sigma}_{\sigma\beta})\frac{\partial}{\partial z^\sigma_\rho} - \\ - \sum_{\substack{\rho,\sigma=1\\(\rho<\sigma)}}^{n} (T^{\beta\sigma*}_{\rho\alpha} - z^{\sigma*}_\rho\, T^{\beta\sigma*}_{\sigma\alpha})\frac{\partial}{\partial z^{\sigma*}_\rho}, \end{aligned} \tag{2.12}$$

$$T^{\alpha\sigma}_{\rho\beta} = Z^{\sigma,\sigma+1,\ldots,n}_{\beta,\sigma+1,\ldots,n}\,\delta^\alpha_\rho + Z^{\sigma,\sigma+1,\ldots,n}_{\rho,\beta,\ldots,n}\,\delta^\alpha_{\sigma+1} + \ldots + Z^{\sigma,\sigma+1,\ldots,n}_{\rho,\sigma+1,\ldots,\beta}\,\delta^\alpha_n, \tag{2.13}$$

$$K^\alpha_\beta = \rho_2\, P^\alpha_\beta + \mathrm{i}\, N^\alpha_\beta, \tag{2.14}$$

$$\begin{aligned} N^\alpha_\beta = -\mathrm{i}\sum_{p=2}^{n}\Bigg[\frac{2(p-1)}{n}\,\delta^\alpha_\beta - \frac{m_p+2(p-1)}{2}\,(T^{\alpha\, p}_{p\,\beta} - T^{\alpha\, p+1}_{p+1\,\beta}) + \\ + \frac{m_p-2(p-1)}{2}\,(T^{\beta\, p*}_{p\,\alpha} - T^{\beta\, p+1*}_{p+1\,\alpha})\Bigg] - \mathrm{i}\sum_{\substack{\rho,\sigma=1\\(\rho<\sigma)}}^{n} (T^{\alpha\sigma}_{\rho\beta} - z^\sigma_\rho\, T^{\alpha\sigma}_{\sigma\beta})\frac{\partial}{\partial z^\sigma_\rho} - \\ - \mathrm{i}\sum_{\substack{\rho,\sigma=1\\(\rho<\sigma)}}^{n} (T^{\beta\sigma*}_{\rho\alpha} - z^{\sigma*}_\rho\, T^{\beta\sigma*}_{\sigma\alpha})\frac{\partial}{\partial z^{\sigma*}_\rho}, \end{aligned} \tag{2.15}$$

$$P^\alpha_\beta = \sum_{p=2}^{n} s_{2p}\,(A_p)^\alpha_\beta, \qquad s_{2p} = \frac{\rho_p}{\rho_2}, \tag{2.16}$$

$$(2.17)\quad \begin{aligned}(A_p)^{\alpha}_{\beta} = &\frac{1}{(n-p)!}\frac{1}{A_{p\,\nu_{p+1},\dots,\nu_n}}\sum_{\nu_{p+1},\dots,\nu_n=1} Z^{p,p+1,\dots,n}_{\beta,\nu_{p+1},\dots,\nu_n} Z^{p,p+1,\dots,n*}_{\alpha,\nu_{p+1},\dots,\nu_n} - \\ &-\frac{1}{(n-p-1)!}\frac{1}{A_{p+1\,\nu_{p+2},\dots,\nu_n}}\sum_{\nu_{p+2},\dots,\nu_n=1}^{n} Z^{p+1,p+2,\dots,n}_{\beta,\nu_{p+2},\dots,\nu_n} Z^{p+1,p+2,\dots,n*}_{\alpha,\nu_{p+2},\dots,\nu_n} - \frac{1}{n}\delta^{\alpha}_{\beta}.\end{aligned}$$

We observe that

$$(2.18)\quad (A_p)^{\alpha}_{\alpha} = (A_p)^1_1 + (A_p)^2_2 + \dots + (A_p)^n_n = 0, \qquad p = 2, 3, \dots, n.$$

Obviously, by the following contraction

$$\lim_{\rho_2\to\infty}\frac{1}{\rho_2}K^{\alpha}_{\beta} = P^{\alpha}_{\beta},$$

we obtain the quantities P^{α}_{β} which, together with J^{α}_{β}'s, generates a class of unitary representations of the Lie algebra of $SU(n)\times) T_{n^2-1}$. The structure equations of this algebra are on the form :

$$(2.19)\quad \begin{cases} [J^{\alpha}_{\beta}, J^{\lambda}_{\mu}] = \delta^{\alpha}_{\mu} J^{\lambda}_{\beta} - \delta^{\lambda}_{\beta} J^{\alpha}_{\mu}, \\ [J^{\alpha}_{\beta}, P^{\lambda}_{\mu}] = \delta^{\alpha}_{\mu} P^{\lambda}_{\beta} - \delta^{\lambda}_{\beta} P^{\alpha}_{\mu}, \\ [P^{\alpha}_{\beta}, P^{\lambda}_{\mu}] = 0. \end{cases}$$

The UIR's of Lie algebra of $SU(n)\times) T_{n^2-1}$, may be written in a $SU(n)$ basis, in which the problem of reduction to the subgroup $SU(n)$ is solved. The advantage of a such solution may be used for $SL(n, C)$ if it is possible to obtain some formulas so that the generators of $SL(n, C)$ can be written as polynomial functions of the generators of suitable chosen unitary representations of $SU(n)\times) T_{n^2-1}$. Such formulas, named the extension formulas [1], [2], gives the possibility to relate the principal nondegenerate series of $SL(n, C)$ with a suitable chosen class of unitary representations of $SU(n)\times) T_{n^2-2}$. In other words, the extension formulas depends, in general, on the considered series of $SL(n, C)$. For example, in the case of the maximally degenerate series of $SL(n, C)$ on applies the Gell-Mann formula [4], while for the principal nondegenerate series it is the formula which will be given bellow.

3. The Extension Formulas. In any unitary representation of $SL(n, C)$ belonging to the principal nondegenerate series the generators K^{α}_{β} can be expressed as functions of $SU(n)\times) T_{n^2-1}$ generators J^{α}_{β} and P^{α}_{β}. (The $SU(n)$ part is carried over unchanged from one group to the other). We refer to these formulas as expansion formulas. To obtain the expansion formulas, we consider the following expressions :

$$(3.1)\quad (Q_i)^{\alpha}_{\beta} = P^{\alpha}_{\gamma_1} P^{\gamma_1}_{\gamma_2} \dots P^{\gamma_{i-2}}_{\beta} - \frac{1}{n}\delta^{\alpha}_{\beta} P^{\lambda}_{\mu_1} P^{\mu_1}_{\mu_2} \dots P^{\mu_{i-2}}_{\lambda}, \quad (i = 2, 3, \dots, n).$$

If we use Eqs. (2.16) and (2.17), then it is easy to see that $(Q_i)^{\alpha}_{\beta}$' s can be written as

$$(3.2) \qquad (Q_i)^{\gamma}_{\beta} = \sum_{p=2}^{n} s_{ip} (A_p)^{\alpha}_{\beta}, \qquad (i = 2, 3, \ldots, n).$$

The coefficients s_{ip} can be computed if the following properties of $(A_p)^{\alpha}_{\beta}$'s are used

$$(3.3) \qquad (A_p)^{\alpha}_{\beta}(A_q)^{\gamma}_{\beta} = \frac{n\delta_{pq} - 2}{2n} [(A_p)^{\gamma}_{\alpha} + (A_q)^{\gamma}_{\alpha}] + \frac{n\delta_{pq} - 1}{n^2} \delta^{\gamma}_{\alpha}.$$

For example, if $i = 2$ then $(Q_2)^{\alpha}_{\beta} = P^{\alpha}_{\beta}$ and Eq. (3.2) is equivalent with (2.16). Then, for $i = 3, 4$ we obtain

$$(3.4) \qquad s_{3p} = \sum_{q=2}^{m} s_{2p} s_{2q} \frac{n\delta_{pq} - 2}{n}, \qquad p = 2, 3, \ldots, n$$

and

$$(3.5) \qquad s_{4p} = \sum_{q,r=2}^{n} s_{2p} s_{2q} s_{2r} \left(\frac{n\delta_{pq} - 2}{2n} \frac{n\delta_{pr} - 2}{n} + \frac{n\delta_{qr} - 2}{n} \frac{n\delta_{pq} - 2}{2n} + \frac{n\delta_{qr} - 1}{n^2} \right).$$

$$p = 2, 3, \ldots, n.$$

Using $(Q_i)^{\alpha}_{\beta}$'s we try to express the generators K^{α}_{β} as linear combinations of J^{α}_{β} and P^{α}_{β} :

$$(3.6) \qquad K^{\alpha}_{\beta} = \rho_2 P^{\alpha}_{\beta} + \mathrm{i} \sum_{j=2}^{n} a_j (Q_j J - J Q_j) + \mathrm{i} \sum_{\substack{j,k=2 \\ (j>k)}}^{n} a_{jk} (Q_j J Q_k - Q_k J Q_j)^{\alpha}_{\beta},$$

where the coefficients a_j and a_{jk} $(j > k)$ are determinated by identification of Eqs. (2.14) and (3.6). In this way, one gets :

$$(3.7) \qquad \sum_{j=2}^{n} s_{jp} a_j = \frac{2(p-1)}{n}, \qquad p = 2, 3, \ldots, n ;$$

$$(3.8) \qquad \sum_{\substack{j,k=2 \\ (j>k)}}^{n} (s_{jp} s_{kq} - s_{jq} s_{kp}) a_{jk} = 1, \quad p, q = 2, 3, \ldots, n, \quad p > q.$$

If $n = 3$ we reobtain the results of [1] for $SL(3, C)$ and if $n = 4$ those of [2] for $SL(4, C)$.

For every n it is necessary to solve the system of equations (3.7) and (3.8) in order to determine the coefficients a_j and a_{jq} . Thus, the extension formulas (3.6) are completely determinated. It remains to identify the parameters that define the $SU(n)\times) T_{n^2-1}$ representation generated by J^{α}_{β} and P^{α}_{β} . This will be taken elsewhere.

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Polytechnic Institute, Jassy,
Department of Physics
and
University „Al. I. Cuza", Jassy

REFERENCES

1. Mukunda N., J. Math. Phys., **10**, 897 (1969).
2. Zet Gh., Bull. Inst. polit. Iaşi, XX (XXIV), *3 – 4*, s. I (1974).

3. Goebel C., Phys. Rev. Lett., **16**, 1130 (1966).
4. Zet G., C. R. Acad. Sc. Paris, **276**, Série *A*, 789 (1973).
5. Naimark M., in *High Energy Physics and Particles Theory*, Kiev, 1967 (in Russian).

FORMULE DE EXTENSIUNE PENTRU $SL(n, C)$: SERIA PRINCIPALĂ NEDEGENERATĂ

(Rezumat)

Se construiesc formulele de extensiune care leagă generatorii infinitezimali ai grupului $SL(n, C)$ în reprezentările din seria principală nedegenerată, de generatorii grupului $SU(n) \times) T_{n^2-1}$ dintr-o clasă de reprezentări unitare convenabil alese. Pentru cazurile particulare $n = 3, 4$ se reobţin rezultatele din lucrările anterioare [1], [2].

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PHYSICAL TETRADS AND A NEW DERIVATION OF SCHWARZSCHILD'S METRIC

BY

CORNELIU D. CIUBOTARIU

1. Introduction. Einstein's equations for the gravitational field are valid for any system of coordinates and make it difficult for one to distinguish physical effects from effects of the curvature of the coordinate system. To avoid this difficulty one has been proposed to obtain a unique definition of the physical system of reference.

Cattaneo (1958) has defined a physical system of reference as a fluid of reference, that is ∞^3 ideal particles whose time tracks are the coordinate lines $x^0 = \text{var}$. The congruence of these coordinate lines determines a unique field $u^\mu \equiv \lambda^\mu_{(0)}$ of unit tangents, whose expression in local coordinates is

$$u^\mu = \left(\frac{1}{\sqrt{-g_{00}}}, 0, 0, 0\right). \tag{1.1}$$

In any point of the space-time this vector may be identified with the 4-velocity of a stationary object in the gravitational field determined by $g_{\alpha\beta}$.

The purpose of the present paper is to provide a unique definition of the observer field associated to a given gravitational field, and to indicate a method of generating the solutions of the Einstein equations without using these equations. These results are obtained by means of a suitable definition of the tetrad field of orthonormal vectors, and with the use of the principle of equivalence.

2. Static Gravitational Field with Spherically Symmetry. Let us consider the general spherically symmetric static line element (we use the signature $(-+++)$):

$$ds^2 = g_{00}\, c^2\, dt^2 + g_{11}\, dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{2.1}$$

For easy reference we give the surviving components of the Christoffel symbols:

$$\begin{aligned} \Gamma^0_{10} &= \frac{1}{2}(\ln g_{00})_{,1}, & \Gamma^1_{11} &= \frac{1}{2}(\ln g_{11})_{,1}, \\ \Gamma^1_{22} &= -\frac{r}{g_{11}}, & \Gamma^1_{33} &= -\frac{r\sin^2\theta}{g_{11}}, \end{aligned} \tag{2.2}$$

$$\Gamma^1_{00} = -\frac{g_{00,1}}{2g_{11}}, \qquad \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \tag{2.2}$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta, \qquad \Gamma^3_{23} = \operatorname{ctg}\theta,$$

where $x^0 = \mathrm{ct}$, $x^1 = r$, $x^2 = \theta$, $x^3 = \varphi$, and partial derivative is denoted by subscript , .

From the invariant Frenet-Serret formulas (S y n g e, 1960) :

$$\frac{\delta\lambda^\mu_{(0)}}{\delta s} b = b\lambda^\mu_{(1)}, \qquad \frac{\delta\lambda^\mu_{(1)}}{\delta s} = c\lambda^\mu_{(2)} + b\lambda^\mu_{(0)}, \tag{2.3}$$
$$\frac{\delta\lambda^\mu_{(2)}}{\delta s} = d\lambda^\mu_{(3)} - c\lambda^\mu_{(1)}, \qquad \frac{\delta\lambda^\mu_{(3)}}{\delta s} = -d\lambda^\mu_{(2)},$$

(1.1), (2.1) and (2.2) we obtain the following tetrad of orthonormal vectors :

$$u^\mu \equiv \lambda^\mu_{(0)} = \left(\frac{1}{\sqrt{-g_{00}}}, 0, 0, 0\right), \qquad n^\mu \equiv \lambda^\mu_{(1)} = \left(0, \frac{1}{\sqrt{g_{11}}}, 0, 0\right), \tag{2.4}$$
$$b^\mu \equiv \lambda^\mu_{(2)} = \left(0, 0, \frac{1}{r}, 0\right), \qquad N^\mu \equiv \lambda^\mu_{(3)} = \left(0, 0, 0, \frac{1}{r\sin\theta}\right),$$

where n^μ, b^μ, and N^μ are the first, second, and third (space-like) unit normal. ((0), (1), (2), and (3) label the vector ; μ, ν,..., $= 0, 1, 2, 3$ are the vector index). The scalar

$$b = \frac{1}{2\sqrt{g_{11}}}(\ln g_{00})_{,1} \tag{2.5}$$

is the first curvature, and $c = d = 0$ are the second and third curvature.

$r = \mathrm{const}$ implies $b = \mathrm{const}$ and hence on each sphere around the field center the coordinate line $x^0 = \mathrm{var.}$ is a hyperbola of constant curvature.

The tetrad (2.4) of orthonormal vectors may be interpreted physically as the local Lorentz frame of an observer whose 4-velocity is the time-like frame vector u^μ and whose Cartesian axes are the space-like frame vector n^μ, b^μ and N^μ respectively. Measurement of a tensor by this observer yield its physical components in this frame. The procedure of taking the components of a tensor with respect to the local tetrad on the world line of an observer is equivalent to transforming the tensor to the instantaneous rest frame of that observer.

Furthermore, the tetrad (2.4) is easily shown to satisfy the Fermi-Walker transport and, therefore, it is this frame of reference which must appear to give the correct relativistic generalization of the Newtonian concepts.

3. The Derivation of the Schwarzschild Metric without the Field Equations. With the use of the principle of equivalence and the conservation of energy in the Newtonian approximation, in 1944 W. L e n z (see S o m m e r f e l d, 1952) has obtained the Schwarzschild metric without the field

equations. Latter, T a n g h e r l i n i (1962), and E r i k s s o n (1963) have derived analogous results completed with additional comments on measurable quantities.

The derivation of Schwarzschild line element presented here differs from that of the preceding authors in the application of the equivalence principle and in the fact that there is no need to justify that an infinitesimal rod oriented transversaly preserves its length unchanged relative to the system of coordinates, as it is done in the works of Lenz, and Eriksson.

Let us consider the central gravitational field of a mass M located at the origin of the spherically coordinate system. To a first approximation it is easily to show from an elementary application of the Doppler effect and the equivalence principle (see, for example, W e b e r, 1961, and T a n g h e r l i n i, 1961) that for a clock at rest (supported clock) in the gravitational field created by M, the proper time is given by

$$ds^2 = -\left(1 - \frac{A}{r}\right) c^2 dt^2 , \tag{3.1}$$

where $A = \frac{2kM}{c^2}$, and k is the gravitational constant. The 4-velocity of the supported clock is given by (1.1) and hence, according to (3.1), we have

$$g_{00} = -\left(1 - \frac{A}{r}\right), \tag{3.2}$$

and

$$u_0 = -\frac{1}{u^0} = -\sqrt{1 - \frac{A}{r}}, \tag{3.3}$$

after having made use of equation $u_\mu u^\mu = -1$.

Next we propose to find g_{11} as a function of r, by use of (3.2). For this purpose we obtain from (2.2) and (2.3) the relation

$$\frac{\delta\lambda^\nu_{(0)}}{\delta s} = \Gamma^\nu_{00} u^0 u^0 = b\lambda^\nu_{(1)} . \tag{3.4}$$

Since $\frac{\delta\lambda^\nu_{(0)}}{\delta s} = a^\nu$, where a^ν is the 4-acceleration vector of the supported observer, for a weak field we may consider that (L a n d a u and L i f s h i t z, 1951)

$$a^\nu = \left(0, \frac{\alpha}{c^2}, 0, 0\right) = bn^\nu , \tag{3.5}$$

where $\alpha = \frac{kM}{r^2}$ is the proper acceleration of an observer gravitationally equivalent to the supported observer.

The only nonvanishing Γ^ν_{00} for the metric (2.1) is Γ^1_{00}, and hence from (2.2) and (3.4) we find

$$\Gamma^1_{00} = -\frac{1}{2}\frac{g_{00,1}}{g_{11}} = \frac{\alpha}{c^2}\left(1 - \frac{A}{r}\right) = \frac{A}{2r^2}\left(1 - \frac{A}{r}\right). \tag{3.6}$$

Substituting (3.2) in (3.6) we get

$$g_{11} = \left(1 - \frac{A}{r}\right)^{-1}. \tag{3.7}$$

Thus we arrive at the Schwarzschild's external solution of Einstein's equations :

$$ds^2 = -\left(1 - \frac{A}{r}\right)c^2\,dt^2 + \left(1 - \frac{A}{r}\right)^{-1} dr^2 + r^2\,(d\theta^2 + \sin^2\theta\,d\varphi^2). \tag{3.8}$$

4. Conclusions. The main conclusions of this work are as follows :

(i) The orthonormal vectors of the tetrad field of supported abservers are the unit tangent to the world line, the first, second and third unit normal.

(ii) The Schwarzschild line element may be obtained from the applications of the Doppler effect, the equivalence principle, and the relativistic generalization of the acceleration vector. The "problem" of the transformation of an infinitesimal rod oriented transversely, in the derivation, does not exist.

(iii) Schwarzschild's exterior solution to Einstein's field equations corresponds to a weak gravitational field.

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Polytechnic Institute, Jassy,
Department of Physics

REFERENCES

1. Cattaneo C., Il Nuovo Cimento, X, *2*, 319 (1958).
2. Eriksson K. E., Arkiv för Physik, **25**, *15*, 167 (1963).
3. Landau L., Lifshitz E., *The Classical Theory of Fields.* Adison-Wesley, Cambridge, Mass., 1951.
4. Sommerfeld A., *Lectures on Theoretical Physics.* Vol. III, New York, 1952.
5. Synge J. L., *Relativity : The General Theory.* North-Holland Publishing Company, Amsterdam, p. 10, 1960.
6. Tangherlini F. R., Il Nuovo Cimento, XXV, *5*, 1081 (1962).
7. Tangherlini F. R., Supplemento Nuovo Cimento, **20**, serie *X*, 1 (1961).
8. Weber J., *General Relativity and Gravitational Waves.* New York, 1961.

TETRADE FIZICE ŞI O NOUĂ DEDUCERE A METRICEI LUI SCHWARZSCHILD

(Rezumat)

Se dă o interpretare fizică a tetradelor vectorilor ortonormali asociaţi metricei statice generale cu simetrie sferică şi se cercetează proprietăţile de transport ale vectorilor. Se propune o nouă deducere a metricei lui Schwarzschild fără utilizarea ecuaţiilor de cîmp ale lui Einstein.

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HEAT OF DIAMAGNETIC CRYSTALS IN MAGNETIC FIELD

BY

GH. MAFTEI and G. POPESCU-BOTEZ

1. Introduction. The magnetic field influences the motion of ions and consequently the thermal oscillations of an ionic, crystalline lattice as well. As in magnetohydrodynamics [2], [4] for conducting fluide, the dispersion equation of the waves propagating in the presence of the magnetic field through the ionic crystal, will be expected to differe fron the dispersion equation corresponding to the absence of this field.

Thus, a deviation may appear in the very spectrum of stationary oscillations of the crystal in the presence of the magnetic field as compared to the spectrum corresponding to the absence of this field. Therefore, in the presence of the magnetic field, the conclusions of [1], [3], [5], [6] concerning the oscillation spectrum of cubic lattices are no longer entirely accurate. To find out the spectrum, the influence of the magnetic field on the ions motion should be considered, too.

This influence has been envisaged in a previous work for a linear lattice in a homogeneous magnetic field, parallel to the lattice [6]. A deviation has been broughtout with respect to the absence of the magnetic field in the low frequency range, the distribution function, for $\nu \to 0$, tending to infinity, instead of staying limited.

A deviation from the oscillation spectrum in the presence of the magnetic field as against the absence of this field is expected to appear also in tridimensional crystalline lattices at the limit of low frequencies. This king of deviations is pointed out in the present paper.

We show that in the presence of the magnetic field the density of the normal vibrations at low frequency is proportional to the frequency and not to the square of the frequency, hence a deviation from Debye law. This deviation is also reflected in the variation of the specific heat with temperature. Since at low temperatures only the low frequency oscillations are actually excited, the influence of the magnetic field on the variation of the specific heat with the temperature for diamagnetical ionic crystals, is expected to appear at the limit of the low temperatures i.e. at such temperatures where $kT < \hbar\omega_H$, where k and $\hbar$ are Boltzmann and Planck constants and ω_H is the angular cyclotronic frequency of the lattice ions.

A critical temperature may be thus introduced

$$T_c = \frac{\hbar\,\omega_H}{k}, \tag{1.1}$$

at wich one should expect the existence of a deviation in the law of variation of the specific heat of ionic diamagnetic crystals as compared to the proportionality $C_v \sim T^3$. At $T < T_c$ temperatures, if we consiser the above mentioned deviation of the oscillation spectrum of the ionic lattice, the proportionality $C_v \sim T^2$ is obtained in the presence of the magnetic field.

Therefore, in the presence of the magnetic field, in the range $T < T_c$, the T^2 law will result, and at $T > T_c$ one should applay the T^3 law.

If we consider for the maximum intensity of available fields $H \sim 10^5$ Oe, for T_c we obtain values of $\sim 10^{-2}$ K.

The present paper will be concerned with the oscillations of the ionic, diamagnetic cubic lattice in a homogeneous magnetic field paralell to an edge of the cube. We shall construct, based on the dispersion equation, the density of the normal vibrations and we shall deduce the $C_v \sim T^2$ law for the already mentioned temperature range.

2. Dispersion Equation of the Oscillations of the simple, cubic, ionic Bravais lattice in the presence of the magnetic field. The following equations may be formulated for the ions oscillations with order N° l, m, n in the cubic crystalline lattice, where we denote the coefficient characteristic of the elastic interaction forces among the ions in the immediate neighbourhood by α, and that characteristic of the interaction among remoter ones by γ:

$$
\begin{aligned}
M\ddot{\zeta}_{l,m,n} &= \alpha(\xi_{l+1,m,n} + \xi_{l-1,m,n} - 2\xi_{l,m,n}) + \gamma(\xi_{l+1,m,n+1} + \xi_{l-1,m,n-1} + \xi_{l+1,m,n-1} + \\
&+ \xi_{l+1,m+1,n} + \xi_{l+1,m-1,n} + \xi_{l-1,m,n+1} + \xi_{l-1,m+1,n} + \xi_{l-1,m-1,n} - 8\xi_{l,m,n} + \eta_{l+1,m+1,n} + \\
&+ \eta_{l-1,m-1,n} - \eta_{l+1\,m-1,n} - \eta_{l-1,m+1,n} + \zeta_{l+1,m,n+1}, + \zeta_{l-1,m,n-1} - \zeta_{l+1,m,n-1} - \zeta_{l-1,m,n+1}) + \\
&+ QB\dot{\eta}_{l,m,n}, \\
M\ddot{\eta}_{l,m,n} &= \alpha(\eta_{l,m+1,n} + \eta_{l,m-1,n} - 2\eta_{l,m,n}) + \gamma(\eta_{l+1,m+1,n} + \eta_{l,m+1,n+1} + \eta_{l,m+1,n-1} + \\
&+ \eta_{l-1,m+1,n} + \eta_{l+1,m-1,n} + \eta_{l,m-1,n+1} + \eta_{l,m-1,n-1} + \eta_{l-1,m-1,n} - 8\eta_{l,m,n} + \xi_{l+1,m+1,n} + \\
&+ \xi_{l-1,m-1,n} - \xi_{l+1,m-1,n} - \xi_{l-1,m+1,n} + \zeta_{l,m+1,n+1} + \zeta_{l,m-1,n-1} - \zeta_{l,m-,n+1} - \\
&- \zeta_{l,m+1,n-1}) - QB\dot{\xi}_{l,m,n}, \\
M\ddot{\zeta}_{l,m,n} &= \alpha(\zeta_{l,m,n+1} + \zeta_{l,m,n-1} - 2\zeta_{l,m,n}) + \gamma(\zeta_{l+1,m,n+1} + \zeta_{l,m+1,n+1} + \zeta_{l-1,m,} + \\
&+ \zeta_{l,m-1,n+1} + \zeta_{l,m-1,n-1} + \zeta_{l+1,m,n-1} + \zeta_{l,m+1,n-1} + \zeta_{l-1,m,n-1} - 8\zeta_{l,m,n} + \xi_{l+1,m,n+1} + \\
&+ \xi_{l-1,m,n-1} - \xi_{l+1,m,n-1} - \xi_{l-1,m,n+1} + \eta_{l,m+1,n+1} + \eta_{l,m-1,n-1} - \eta_{l,m+1,n-1} - \eta_{l,m-1,n+1}).
\end{aligned}
\tag{2.1}
$$

We have denoted by Q the charge of the ion and by B the magnetic induction.

The oscillations of the lattice may be considered as an overlapping of the harmonic oscillations. For a single normal oscillation one may write

$$
\xi_{l,m,n} = e^{-i(wt-\mathbf{kr}_{l,m,n})}, \quad \eta_{l,m,n} = \eta e^{-i(wt-\mathbf{kr}_{l,m,n})}, \quad \zeta_{l,m,n} = \eta e^{-i(wt-\mathbf{kr}_{l,m,n})} \tag{2.2}
$$

if we take into account that $\mathbf{r}_{l,m,n} = l\mathbf{i} + m\mathbf{j} + n\mathbf{k}$ are the vector radii of the lattice points, from (2.1) and (2.2) we have the system

$$
\begin{aligned}
&[(1 - \cos k_x)\,\omega_0^2 + (2 - \cos k_x \cos k_z - \cos k_x \cos k_y)\,\omega_1^2 - \omega^2]\,\xi - \\
&- (\cos k_x \cos k_y\, \omega_1^2 - i\,\omega\omega_H)\,\eta - \cos k_x \cos k_z\, \omega_1^2\, \zeta = 0, \\
&- (\cos k_x \cos k_y\, \omega_1^2 + i\,\omega\omega_H)\,\xi + [(1 - \cos k_y)\,\omega_0^2 + (2 - \cos k_y \cos k_x - \\
&- \cos k_y \cos k_z)\,\omega_1^2 - \omega^2]\,\eta - \cos k_y \cos k_z\, \omega_1^2\, \zeta = 0, \\
&- \cos k_x \cos k_z\, \omega_1^2\, \xi - \cos k_y \cos k_z\, \omega_1^2\, \eta + [(1 - \cos k_z)\,\omega_0^2 + (2 - \\
&- \cos k_x \cos k_z - \cos k_y \cos k_z)\,\omega_1^2 - \omega^2]\,\zeta = 0,
\end{aligned}
\tag{2.3}
$$

where

$$
\omega_0^2 = \frac{2\alpha}{M}, \quad \omega_1^2 = \frac{4\gamma}{M} \quad \text{and} \quad \omega_H = \frac{QB}{M}.
$$

Taking account of the fact that at a cubic crystal, with N ions along an edge one may write conditions of the form $\xi_l = \xi_{l+N}$, it results that the wave vector $\mathbf{k}$ may have only discrete values with the components

$$k_x = \frac{2\pi}{N} p, \qquad k_y = \frac{2\pi}{N} q, \qquad k_z = \frac{2\pi}{N} r, \tag{2.4}$$

where p, q and r are whole nombers.

Using notation (2.4), the dispersion equation results from the compatibility of the system (2.3)

$$\begin{vmatrix} \left(1-\cos\frac{2\pi}{N}p\right)\omega_0^2 + \left(2-\cos\frac{2\pi}{N}p\cos\frac{2\pi}{N}q - \cos\frac{2\pi}{N}p\cos\frac{2\pi}{N}r\right)\omega_1^2 - \omega^2 & \sin\frac{2\pi}{N}p\sin\frac{2\pi}{N}q\,\omega_1^2 + i\,\omega\omega_H & \sin\frac{2\pi}{N}p\sin\frac{2\pi}{N}r\,\omega_1^2 \\ \sin\frac{2\pi}{N}p\sin\frac{2\pi}{N}q\,\omega_1^2 - i\,\omega\omega_H & \left(1-\cos\frac{2\pi}{N}q\right)\omega_0^2 + \left(2-\cos\frac{2\pi}{N}p\cos\frac{2\pi}{N}q - \cos\frac{2\pi}{N}q\cos\frac{2\pi}{N}r\right)\omega_1^2 - \omega^2 & \sin\frac{2\pi}{N}q\sin\frac{2\pi}{N}r\,\omega_1^2 \\ \sin\frac{2\pi}{N}p\sin\frac{2\pi}{N}r\,\omega_1^2 & \sin\frac{2\pi}{N}q\sin\frac{2\pi}{N}r\,\omega_1^2 & \left(1-\cos\frac{2\pi}{N}r\right)\omega_0^2 + \left(2-\cos\frac{2\pi}{N}p\cos\frac{2\pi}{N}r - \cos\frac{2\pi}{N}q\cos\frac{2\pi}{N}r\right)\omega_1^2 - \omega^2 \end{vmatrix} = 0. \tag{2.5}$$

Because of $\omega_0^2 \gg \omega_1^2$ and $\omega_0^2 \gg \omega_H^2$ in the first approximation according to the variable ω_1^2 and ω_H^2, it results from (2.5) the following solutions :

$$\omega_{(1)}^2 = \left(1-\cos\frac{2\pi}{N}r\right)\omega_0^2 + \left(2-\cos\frac{2\pi}{N}p\cos\frac{2\pi}{N}r - \cos\frac{2\pi}{N}q\cos\frac{2\pi}{N}r\right)\omega_1^2,$$

$$\omega_{(2)}^2 = \left(1-\cos\frac{2\pi}{N}p\right)\omega_1^2 + \left(2-\cos\frac{2\pi}{N}p\cos\frac{2\pi}{N}q - \cos\frac{2\pi}{N}p\cos\frac{2\pi}{N}r\right)\omega_1^2 + \frac{1-\cos\frac{2\pi}{N}p}{\cos\frac{2\pi}{N}q - \cos\frac{2\pi}{N}p}\,\omega_H^2,$$

(2.6 a)

$$\omega_{(3)}^2 = \left(1-\cos\frac{2\pi}{N}q\right)w_0^2 + \left(2-\cos\frac{2\pi}{N}p\cos\frac{2\pi}{N}q - \cos\frac{2\pi}{N}q\cos\frac{2\pi}{N}r\right)\omega_1^2 + \frac{1-\cos\frac{2\pi}{N}q}{\cos\frac{2\pi}{N}p - \cos\frac{2\pi}{N}q}\,\omega_H^2,$$

$$\text{if} \quad \cos\frac{2\pi}{N}p \neq \cos\frac{2\pi}{N}q;$$

$$\omega_{(2)} = \sqrt{1 - \cos\frac{2\pi}{N}p}\;\omega_0 + \frac{1}{2}\omega_H, \qquad \omega_{(3)} = \sqrt{1 - \cos\frac{2\pi}{N}p}\;\omega_0 - \frac{1}{2}\omega_H, \tag{2.6b}$$

$$\text{if} \quad \cos\frac{2\pi}{N}q = \cos\frac{2\pi}{N}p,$$

(in the approximation up to the first order according to ω_1).

The deviation from the dispersion equation without the magnetic field is due to the term which contains cyclotronic frequency.

3. Qualitative Study of the influence of the term depending from the intensity of the magnetic field. If we assume ω_1 negligible as to ω_0 and ω_H, equations (2.6a) are simplified and may be written as :

$$\omega_{(1)}^2 = \left(1 - \cos\frac{2\pi}{N}r\right)\omega_0^2,$$

$$\omega_{(2)}^2 = \left(1 - \cos\frac{2\pi}{N}p\right)\omega_0^2 + \frac{1 - \cos\frac{2\pi}{N}p}{\cos\frac{2\pi}{N}q - \cos\frac{2\pi}{N}p}\,\omega_H^2, \tag{3.1}$$

$$\omega_{(3)}^2 = \left(1 - \cos\frac{2\pi}{N}q\right)\omega_0^2 - \frac{1 - \cos\frac{2\pi}{N}q}{\cos\frac{2\pi}{N}q - \cos\frac{2\pi}{N}p}\,\omega_H^2.$$

By representing graphically the intersection of surfaces $\omega_{(2)}^2 = \text{const.}$ and $\omega_{(3)}^2 = \text{const.}$ with the plane pOq in the range $0 < p < N/2$ and $0 < q < N/2$ we get the following :

i) if $\omega_H \to 0$, the intersections become the straight lines represented by dashed lines in Fig. 1. (The vertical straight line corresponds to $\omega_{(2)} = \text{const.}$ and the horizontal one to $\omega_{(3)} = \text{const.}$);

ii) if $\omega_H \neq 0$ there appears a deviation from the straight lines represented by dashed lines. The deviation is only a small correction where $p/q \ll 1$ or $p/q \gg 1$. If $p \approx q$, the approximation on whose basis relations (3.1) and (2.5) have been obtained, is no longer correct. Instead of the last two relations of the system (3.1) we get for $p=q$ the last two relations from (2.6). Therefore in the vecinity of the diagonal $p = q$ the correction related to ω_H in the expression of ω is no longer of the second order, but of the first order.

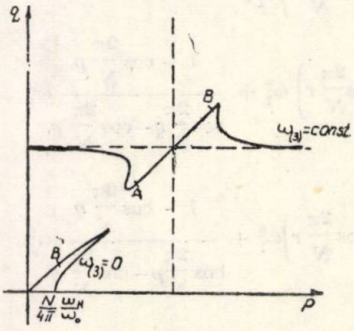

Fig. 1

Consequently the curves corresponding to the intersection of the surfaces $\omega_{(3)}^2 = \text{const.}$ with the plane pOq will be curves similar to those in Fig. 1 represented by solid lines. The appearance of the deviation in the neighbourhood of the diagonal results in the following consequences :

The total number of states inside the domains obtained by the intersection of the surfaces $\omega_{(1)}^2 = \text{const.}$, $\omega_{(2)}^2 = \text{const.}$ and $\omega_{(3)}^2 = \text{const.}$ is changing as a function of the intensity of the magnetic field, because, due to the appearance of distorsions, the volume of these domains is changing too.

The variation of the number of states at low frequency may be appreciated qualitatively as a function of frequency. If no magnetic field would exist, according to (3.1), at the limit of low frequency, we had the proportionalities $r \sim \omega_{(1)}$, $p \sim \omega_{(2)}$ and $q \sim \omega_{(3)}$, therefore the volume in the space of wave vectors, where there are points corresponding to the states with frequencies lower than $\omega = \omega_{(1)} = \omega_{(2)} = \omega_{(3)}$, is proportional to ω^3, and the number of states with the frequency between ω and $\omega + d\omega$ is proportional to the derivatives of the volume limited by the surfaces $\omega = \omega_{(1)} = \omega_{(2)} = \omega_{(3)}$ related to ω. Therefore, for the density of the normal vibrations, we actually get the proportionality with ω^2 corresponding to Debye theory.

However, in the presence of the magnetic field, at the limit of low frequency peculiarity will appear. For $\omega = \omega_{(1)} = \omega_{(2)} = \omega_{(3)} \to 0$ the A intersection point of those surfaces $\omega = \text{const.}$ is moving to negative domains of the phase space and point B remains on the diagonal $p = q$, having, on the basis of the last relation in the system (2.6) at the limit, the coordinate

$$p = \frac{N}{4\pi} \frac{\omega_H}{\omega_0}. \tag{3.2}$$

Considering that $\omega_{(3)} = 0$ in (3.1) it results equation

$$\cos \frac{2\pi}{N} q - \cos \frac{2\pi}{N} p = \frac{\omega_H^2}{\omega_0^2}; \tag{3.3}$$

for the intersection of the surface $\omega_{(3)} = 0$ with the plane pOq we get a curve with the same shape as in Fig. 1. The long shape of surfaces closed by curves $\omega_{(3)} = \text{const.}$ is maintained also for smaller $\omega_{(3)}$ values.

The intersections of the surfaces $\omega_{(2)} = \text{const.}$ with the plnae pOq are obtained from those represented in Fig. 1 by reflection with respect to the diagonal $p = q$. Hence, at the limit of low frequency, the domain corresponding to frequencies smaller than $\omega = \omega_{(2)} = \omega_{(3)}$ will become longer towards the diagonal with lengths of the order of the $\frac{\sqrt{2}N}{4\pi} \frac{\omega_H}{\omega_0}$ value, and the width proportional to ω, therefore the area is also proportional to ω. The volume with the points corresponding to frequency states smaller than $\omega = \omega_{(1)}$ towards the axis Or is limited by the plane $r = \text{const.}$ Since at the limit of low frequency $r \sim \omega_{(1)}$, and the area of the surface from the pOq plane is also proportional to ω, the volume corresponding to low frequencies in the space of wave vectors in the presence of the magnetic field, will be proportional to ω^2. Considering that the density of the normal vibrations is the derivative of this volum with respect to the frequency, for the density of the normal vibrations we get that in the presence of the magnetic field, at low frequencies, this should be proportional to ω.

Taking account of the fact that based on Einstein-Bose distribution, the total energy of the oscillations may be written as

$$E = \int \frac{\hbar\omega\, g(\omega)}{e^{\frac{\hbar\omega}{kT}} - 1}\, d\omega \tag{3.4}$$

for the domain where the density of the normal vibrations $g(\omega)$ is proportional to ω from (3.4) the proportionality

$$E \sim T^3 \tag{3.5}$$

will result, i.e. for the specific heat we get at low temperatures, in the presence of the magnetic field, the proportionality

$$C_v \sim T^2. \tag{3.6}$$

It is obvious from (3.4) that (3.6) is satisfied when only frequencies lower than ω_H are actually excited, for which we may use the obove approximation for $g(\omega)$. Hence, condition (1.1) results from (3.4).

4. Spectrum of the Oscillations of the simple, ionic, cubic crystalline lattice in the presence of the magnetic field. By dividing the space of the wave vectors for the range $0 < p < N/2$, $0 < q < N/2$, $0 < r < N/2$ into 27 000 cells, G. Popescu-Botez has calculated the 81 000 numerical values of the functions (2.6) by means of the electronic computer for $\omega_H^2 = \omega_1^2 = 0{,}05\ \omega_0^2$. On the basis of these values Gh. Maftei has constructed the intersections of the surfaces $\omega_{(3)} = \text{const.}$ with the plane pOq, as well as the spectrum of the lattice oscillations. These are represented on Figs. 2—4 and, as it ensues from the following, they are in perfect agreement with the theoretical results previously.

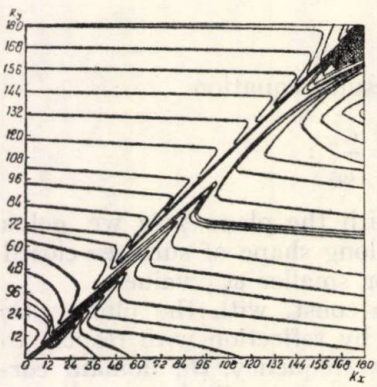

Fig. 2

Fig. 2 shows that curves $\omega_{(3)} = \text{const}$, actually correspond to those represented in Fig. 1. Some shifts are appearing only due to the value of ω_1, that is different from zero.

Fig. 3 represents the appearance probability of an oscillation with the frequency between ω and $\omega + d\omega$ with respect to ω^2. The curve a corresponds to $\omega_{(1)}^2$, curve b to $\omega_{(2)}^2$ and $\omega_{(3)}^2$, and curve c to the resulting probability. The probability resulting in the neighbourhood of the value $\omega = 0$ may be approximated with respect to ω^2 by an oblique straight line, i.e. for small ω values we get

$$w = A + B\omega^2. \tag{4.1}$$

Considering that in Fig. 3 the independent variable is ω^2, the number of oscillations between ω and $\omega + d\omega$ may be written as

$$dN_\omega = w d\omega^2 = (A + B\omega^2)\, d\omega^2, \quad dN_\omega = 2\omega\,(A + B\omega^2)\, d\omega. \tag{4.2}$$

Therefore the density of the normal vibrations may be written as

$$g(\omega) = 2\omega(A + B\omega^2), \tag{4.3}$$

being proportional, at low frequency, to ω; this corresponds to the previous conclusions.

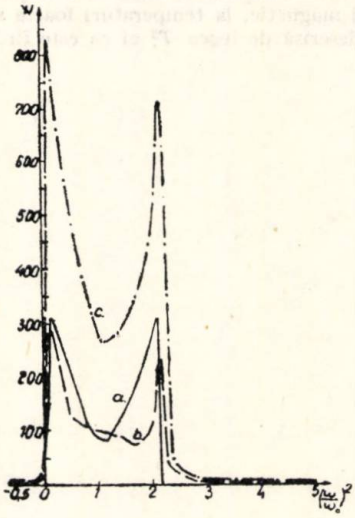

Fig. 3

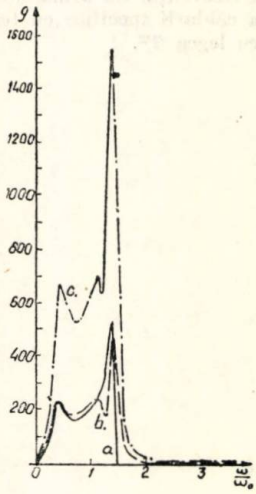

Fig. 4

An additional confirmation is the fact that in the case of the resulting density of the normal vibrations obtained on the basis of the **81 000** numerical values, the tangent to the curve $g = g(\omega)$ in point $\omega = 0$ is an oblique.

The density of the normal vibrations $g = g(\omega)$ is represented in Fig. 4. Curve *a* corresponds likewise to $\omega_{(1)}$, curve *b* to $\omega_{(2)}$ and $\omega_{(3)}$, and curve *c* to all the oscillations of the lattice.

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Polytechnic Institute, Jassy,
Department of Physics

REFERENCES

1. Blackman M., Proc. Roy. Soc. London, **A 159**, 416 (1937).
2. Denisse J., Delcroix F. L., *Théorie des ondes dans les plasmas.* Dunod, Paris, 1961.
3. Fine P. C., Phys. Rev., **56**, 355 (1939).
4. * * * *Handbuch der Physik.* B. 7, T. *1*, Springer-Verlag, Berlin, 1955.
5. Landau L. D., Lifşiţ E. M., *Electrodinamica mediilor continue* (transl. from Russian). Ed. tehn., Buc., 1968.
6. Maftei Gh., Bul. Inst. polit. Iaşi, **XVI (XX)**, *1 – 2*, 59 (1970).

CĂLDURA SPECIFICĂ A CRISTALELOR DIAMAGNETICE ÎN CÎMPURI MAGNETICE

(Rezumat)

Autorii arată că datorită mişcării ciclotronice a ionilor din reţeaua cristalină, spectrul oscilaţiilor de frecvenţă joasă se schimbă iar densitatea vibraţiilor normale devine proporţională cu frecvenţa. Ca urmare, în prezenţa cîmpului magnetic, la temperaturi foarte scăzute, variaţia căldurii specifice cu temperatura nu este descrisă de legea T^3 ci ea este în concordanţă cu legea T^2.

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MECANICĂ TEORETICĂ, FIZICĂ

C- D. 538.2

SUR LES EFFETS D'AUTOCONCENTRATION DANS LES FERROFLUIDES

PAR

EMIL LUCA, V. BĂDESCU, C. COTAE et R. RABINOVICI

1. Introduction. Le dernier temps, de nombreuses études sur les ferrofluides ont traité spécialement leur double caractère, de liquide et de matériel magnétique. Ainsi, présente intérêt l'étude des effets de nonuniformisation spatielle et temporelle de la fraction volumétrique du matériel magnétique solide en suspension.

Quoique les ferrofluides sont définis par Neuringer et Rosensweig comme des suspensions colloïdales ultrastables [1], la pratique de laboratoire a mis en évidence d'effets d'autoconcentration de ceux-ci. Bibik et ses collaborateurs [2] ont observé par voie optique d'effets d'agglomération réversible dans des ferrofluides très dilués, fait important pour la réalisation des dispositifs à moduler l'intensité des faisceaux lumineux. Peterson et Krueger [3] ont observé d'effets d'agglomération irréversible dans le cas des ferrofluides à base de l'eau, fait qui limite le domaine d'aplicabilité de ces derniers.

Dans l'ouvrage présent on donne quelques résultats concernant les effets d'autoconcentration dans d'autres ferrofluides, de différents fractions volumétriques et liquides de base.

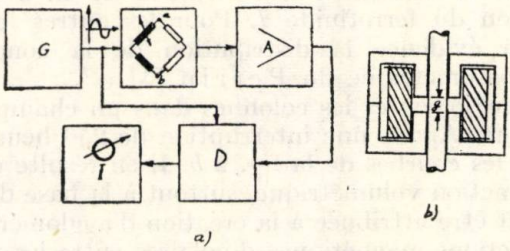

Fig. 1 — Le schéma principal de l'installation de mesure.

2. La technique expérimentale. Pour mesurer les concentrations volumiques dans les ferrofluides étudiés, on a réalisé une installation (fig. 1), fondée sur la proportionalité entre leur perméabilité et la fraction volumétrique du matériel magnétique en suspension.

Les ferrofluides testés ont été introduits dans des tubes en verre longs de 100 mm et de 8 mm diamètres intérieurs. Ces colonnes ont constitué le noyau mobil vérticalement d'un traducteur inductif appartenant à un pont de c.a. (fig. 1 *a*). Le traducteur inductif est un solénoïde à noyau pot, ayant les colonnes intérieures ainsi réalisées qu'il est possible d'obtenir un entrefer haut de 2 mm. Ce circuit magnétique s'enferme par une zonne limitée de la colonne en ferrofluide (fig. 1 *b*), la fraction volumétrique étant mesurée sut cette portion ci. Le deséquilibre du pont c.a. dépend de la fraction volumétrique du matérial magnétique existant dans le ferrofluide situé dans l'ouverture du traducteur.

On a étaloné l'installation à l'aide des épreuves en ferrofluide à fraction volumétrique connue par des mesures de l'aimantation à la saturation.

3. Données expérimentales et discussions. On a étudié des ferrofluides préconisés à être utilisés pour l'étanchement dynamique des machines-outils et dans les systèmes d'amortissement liquide. On trouve leur présentation générale dans le tableau 1.

Tableau 1

No. crt.	Liquide de base	Agent stabilisant	Aimantation à la saturation M_s, A/m
1	Dioctil adipat	Acide oléïque	8.10^3
2	Huile de transformateur	Acide oléïque	$8,10^3$
3	Huile de transformateur	Acide oléïque	16.10^3
4	Pétrole	Acide oléïque	8.10^3
5	Pétrole	Acide oléïque	16.10^3

L'étude comprend les étapes suivantes :

I) Le tracement des formes des fractions volumétriques sur l'hauteur des colonnes pour chaque ferrofluide. Ces profils correspondent aux courbes *1*, *2*, *3*, *4* et *5* de la fig. 2 *a*. Après 250 heures on a tracé de nouveau les profils des mêmes fractions volumétriques et on a constaté une très faible sédimentation du ferrofluide *1*. Pour les autres ferrofluides on n'a pas pu mêtre en évidence la distribution de la concentration conformément à la loi barométrique du Perrin [4].

II) On a placé 5 heures les colonnes dans un champ magnétique uniforme de 8.10^5 A/m. Après une interruption de 250 heures, on a tracé les profils décrits par les courbes de la fig. 2 *b*. Il en résulte une évidente non-uniformité de la fraction volumétrique, surtout à la base des colonnes. Cette sédimentation peut être attribuée à la création d'agglomérats, par le renforcement des intéractions magnétiques dipolaires entre les particules magnétiques, dans le cas de leur alignement dans un champ magnétique. On a démontré théoriquement qu'on peut former d'agglomérats en chaînes, en anneaux ou en grappes, en fonction des dimensions des particules en suspension ainsi que de la nature du solvent.

III). Au début de la deuxième étape, on a créé dans le haut des colonnes un gradient de champ magnétique de 2.10^8 A/m^2. Les profils tracés

immédiatement après l'extraction des colonnes du champ magnétique correspondent aux courbes de la fig. 2 *c*. Il en résulte une concentration du matériel magnétique dans le haut des colonnes due à l'entraînement des agglomérats formés en présence du champ magnétique vers la zone à champ maximum. Les épreuves ainsi obtenues ont été mis à part pendant 250

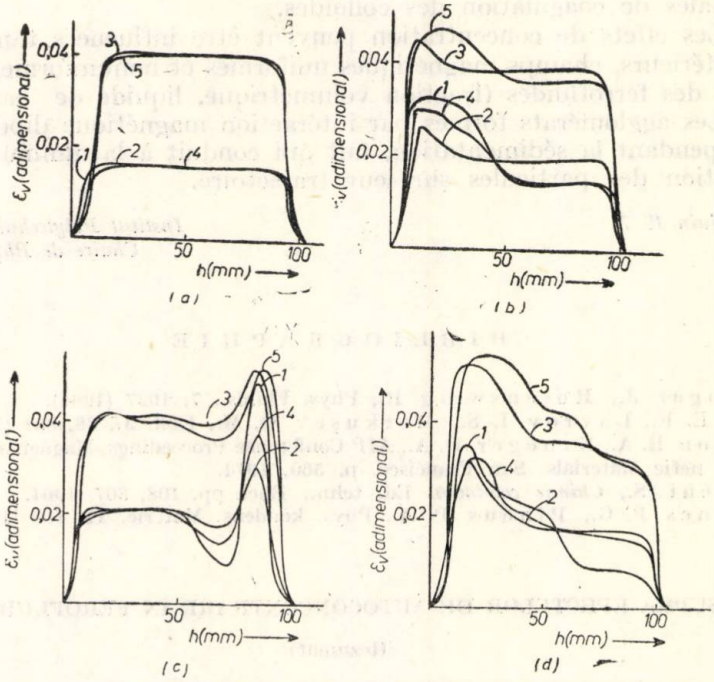

Fig. 2 — Les profils spatiaux des fractions volumétriques $\varepsilon_V = \Sigma v_i / V$ du matériel magnétique solide en suspension, pour les ferrofluides étudiés.

heures, après quoi on a tracé les profils correspondants aux courbes de la fig. 2 *d*. On peut observer que les paquets d'agglomérats ont descendu à cause de la gravitation et, ce qu'est important, une fraction de la totalité des particules magnétiques en suspension a été entraînée dans leur descente, fait confirmé par la diminution de la concentration des particules après la sédimentation des agglomérats vis-à-vis de celle qui précédait la sédimentation.

En comparant les quatre groupes de courbes de la fig. 2, on observe que les effets signalés dépendent tant de la nature des ferrofluides que de leur concentration initiale. L'intéraction magnétique dipolaire entre les particules en suspension dépend du degré d'absorbtion des molécules de l'agent dispersif et du solvent sur celles-ci. Ainsi, dans la I-ère étape, il est évident que le ferrofluide à base d'octil-adipat est moins stable que les deux autres.

Dans la fig. 2*b* on voit que les effets d'agglomération en présence d'un champ magnétique sont plus considérables pour les ferrofluides moins concentrés. La même chose peut être observée dans les courbes de la fig. 2 *c*.

4. Conclusions. 1. Quoique les ferrofluides sont définis comme des suspensions colloïdales ultrastables des particules magnétiques solides dans différents liquides de base, il est évident qu'ils se conduisent d'après les lois normales de coagulation des colloïdes.

2. Les effets de concentration peuvent être influencés tant par des agents extérieurs, champs magnétiques uniformes et nonuniformes, que par la nature des ferrofluides (fraction volumétrique, liquide de base).

3. Les agglomérats formés par intéraction magnétique dipolaire augmentent pendant la sédimentation, fait qui conduit à la diminution de la concentration des particules sur leur trajectoire.

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BIBLIOGRAPHIE

1. Neuringer J., Rosensweig R., Phys. Fluids., 7, 1927 (1964).
2. Bibik E. E., Lavrov I. S., Merkuşev O. M., Koll. J., 28, 631 (1966).
3. Peterson B. A., Krueger D. A., AIP Conference Proceedings, Magnetism and Magnetic Materials, San Francisco, p. 560, 1974.
4. Tilenschi S., *Chimie coloidală*. Ed. tehn., Buc., pp. 108, 367, 1964.
5. de Gennes P. G., Pincus P. A., Phys. kondens. Materie, 11, 189 (1970).

ASUPRA EFECTELOR DE AUTOCONCENTRARE ÎN FEROFLUIDE

(Rezumat)

În prezenta lucrare sînt prezentate comparativ rezultatele studiilor asupra efectelor de autoconcentrare în cîteva ferofluide de interes practic. S-a constatat dependenţa acestor efecte de natura ferofluidelor şi de concentraţia lor iniţială, precum şi influenţarea lor de către factori externi, cum sînt cîmpurile uniforme şi neuniforme.

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RECHERCHES SUR L'EMPLOI DE CERTAINS TYPES D'ÉLECTRODES ET SUR LES ÉCHANTILLONS DE PÂTE DANS L'ANALYSE SPECTROGRAPHIQUE DES ÉMISSIONS DES MINÉRAUX

PAR

GH. TACU

1. Introduction. Dans l'analyse spectrographique d'émission des traces de certains éléments d'un mineral, qui représente un complex de sels, oxides, éléments sous forme d'inclusions et d'autres combinaisons chimiques, il apparaît des phénomènes multiples qui ont lieu dans le plasma de la décharge et qui conduisent à une certaine émission des lignes spectrales.

Ces phénomèns sont conditionés par la nature des essais analysés, du type de la décharge, de la forme et de la nature des électrodes de l'atmosphère dans laquelle a lieu la décharge etc.

Pour l'analyse des quantités très petites des éléments d'un minéral on utilise d'habitude la décharge par arc électrique de courant continu. Ce type de décharge donne une grande sensibilité pour déceler les traces de certains éléments, mais elle est très influencée par les processus qui ont lieu spécialement au moment de l'entrée des éléments dans le plasma de la décharge.

L'arc de courant continu présente en général des fluctuations, qui mènent à une précision diminuée dans le cadre d'une analyse quantitative.

Pour l'amélioration des conditions de l'analyse on a réalisé divers types d'électrodes, divers types de décharge, diverses méthodes pour introduire l'échantillons dans la décharge.

Dans ce travail on présente une série de résultats obtenus par l'introduction des nouveaux types d'électrodes et par l'emploi d'échantillons sous forme de pâte, dans le cadre de l'analyse spectrografique de certains éléments d'un minéral.

La forme et la grandeur de la cavité de l'électrode où on introduit l'échantillon influence les phénomènes de pénétration des éléments dans le plasma des décharges, les phénomènes de volatilisation sélective, la température des électrodes.

D'autre part au moment de l'amorçage de la décharge grâce à certaines variations instantanées du champ électrique, une quantité variable d'échantillons sous forme de poudre est projeté en dehors de la décharge.

En visant une amélioration des conditions d'analyse, on a réalisé quatre types d'électrodes de charbon et une nouvelle méthode de préparation des échantillons.

2. Réalisation des formes d'électrodes et des echantillons de pâte. Vis à vis des deux types classiques d'électrodes 1A et 1B (fig. 1) on a réalisé quatre nouveaux types d'électrodes, c'ést à dire : 2A, 2B, 3A, 3B.

Le volume des cavités exécutées dans les électrodes a été le même de 19 mm³ et la profondeur 2 mm.

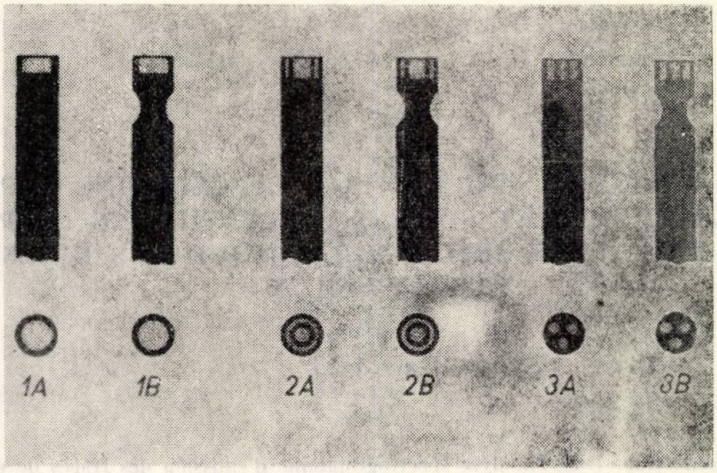

Fig. 1

Ces formes de cavités ont été réalisées à l'aide de mandrins munis de forets et de couteaux spéciaux, que nous présentons dans la fig. 2. Le foret *1* avec la tête de mandrin C_1 produit le type d'électrode 1A ; le système de foret avec couteau *2* et la tête de mandrin C_1 produit le type 2A ; le foret *3* avec tête de mandrin C_2 produit le type 3A. Les types 1B, 2B, 3B sont réalisés de la même façon. Une opération préalable avec un dispositif simple réduit la section sous la cavité de l'électrode.

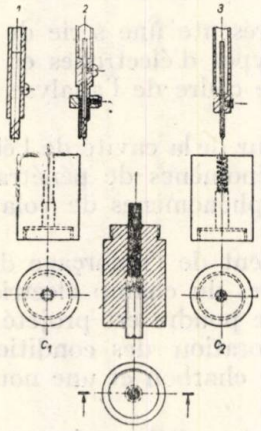

Fig. 2

Avec ces dispositifs on assure une bonne réproduction de ces formes d'électrodes.

L'analyse a été effectuée sur un minéral de pyrite de notre pays, qui présentait de grandes fluctuations pendant sa combustion. Des échantillons de ce minéral ont été introduits tant sous forme de poudre que sous forme de pâte dans la cavité des électrodes.

La poudre a été obtenue par le concassage du minéral dans un mortier d'agate, jusqu'à une granulation moyenne de 3 μm.

La pâte a été obtenue en mélangeant de la poudre avec de l'eau bidistillé en certaines proportions. Pour le minéral recherché, pour 1 g de poudre on a aditionné à l'aide d'une micropipette 0,18 ml d'eau bidistillée.

La pâte d'une certaine consistance, a été introduite dans la cavité de l'électrode avec une spatule de quartz.

Le système d'électrodes avec les échantillons sous forme de poudre et de pâte ont été soumis à un traitement thermique pendant une heure, dans un four à la température de 120°C. Après cet intervalle de temps la pâte se solidifie, formant corps commun avec l'électrode.

Le degré de solidification de la pâte dépend de la nature du minéral et de la grosseur des granulles de la poudre.

On a vérifié que chaque type d'éléctrode contient la même quantité d'échantillon.

L'introduction des échantillons sous forme de poudre par pression dans la cavité de l'électrode est chargée de grandes erreurs de masse. À l'introduction des échantillons sous forme de pâte, on a constaté que dans chaque cavité d'électrode indifféremment de sa forme, entre la même masse. Dans ce cas on peut éliminer les opérations difficiles de pesage de chaque echantillon (comme il est nécessaire dans le cas des poudres).

3. Appareillage et matériaux employés. Les expériences ont été effectuées avec un spectrographe type F.D-S Row—Rathenow, R.D.G., avec trois prismes, ayant le champ spectral entre 3 850 Å et 6 000 Å, et une dispersion variant entre 3,5 Å/mm pour une longeure d'onde de 5 700 Å.

Comme type de décharge on a employé l'arc électrique pour courant continu à la tension de 120 V, produit entre deux électrodes de charbon. On a maintenu un courant de 6 A pour une distance constante entre électrodes de 4 mm.

Les électrodes utilisés ont été des charbons spectrales de type S.U. (R.S.C.) avec le diamètre de 5 mm.

Comme récepteurs de radiation on a employé des plaques spectrales de type ORWO WP-1.

L'analyse spectrographique qualitative a été effectuée avec un spectroprojeteur type SP-2 et la photomesuration des plaques avec un microphotomètre MF-2 Karl Zeiss—Jena.

L'enregistrement des spectres a été fait dans les mêmes conditions pour tous les types d'électrodes et d'échantillons.

4. Résultats des expériences. Le minéral étudié contenant des traces de gallium, indium et thallium, on a observé la reproduction des noircissements des lignes de ces éléments dans les conditions données.

Dans les fig. 3—11 sont présentées les diagrammes de reproduction pour Ga — 4 172, 056 Å, In — 4 511, 323 Å et Tl — 5 350,46 Å, en utilisant les quatres types d'électrodes avec poudre et pâte, prenant dans l'abscisse le nombre des échantillons et dans l'ordonnée les noircissements correspondants.

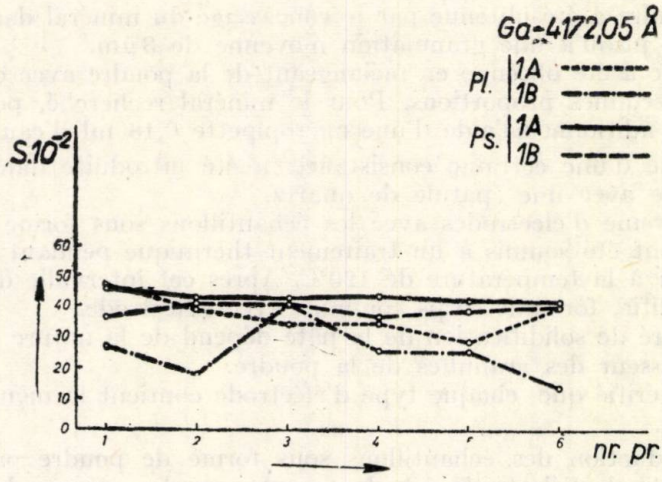

Fig. 3

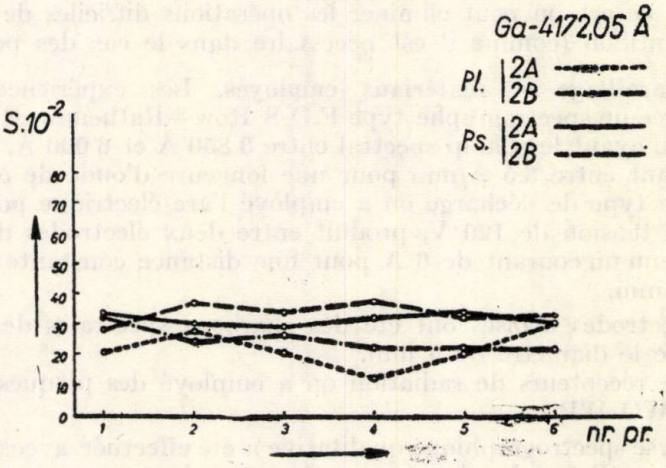

Fig. 4

On observe en géneral une reproduction meilleure, pour les lignes des éléments étudiés, pour les types d'électrodes 2A et 3A avec poudre et un amélioration particulière de la reproduction des noircissements des lignes pour tous les types d'électrodes quand on utilise des échantillons sous forme de pâte.

Les erreurs relatives calculées par rapport à la valeure moyenne des noircissements sont plus petites dans le cas des échantillons avec pâte que dans le cas des échantillons sous forme de poudre (tableau 1).

Tableau 1

Nr.	Elément	Type d'électrode avec du poudre	Erreur relative ±%	Type d'électrode avec de la pâte	Erreur relative ±%
1	Ga–4172 Å	1A	4,3	1A	1,69
		1B	15,0	1B	2,1
		2A	12,8	2A	2,2
		2B	7,0	2B	4,0
		3A	6,3	3A	1,9
		3B	5,7	3B	4,1
2	In–4511 Å	1A	2,0	1A	1,29
		1B	6,3	1B	2,4
		2A	2,3	2A	1,0
		2B	2,8	2B	1,9
		3A	5,0	3A	1,1
		3B	3,84	3B	1,4
3	Tl–5350 Å	1A	1,1	1A	0,47
		1B	7,0	1B	1,06
		2A	0,8	2A	0,61
		2B	1,9	2B	0,93
		3A	1,8	3A	0,91
		3B	2,2	3B	1,3

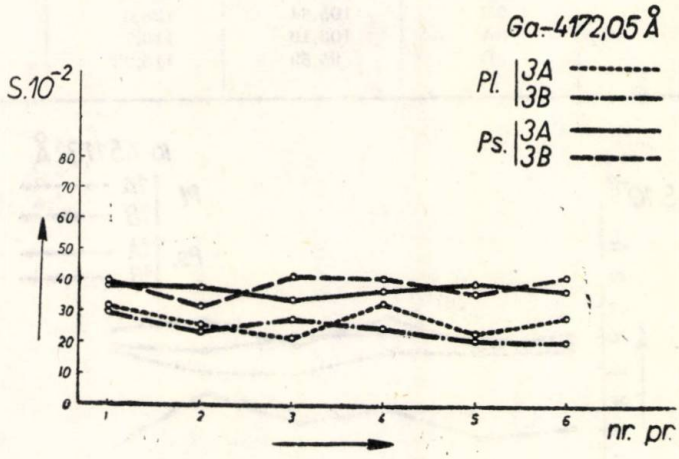

Fig. 5

Un résultat particulier obtenu par l'introduction des échantillos de pâte, est l'intensification des lignes spectrales de certaines éléments parmi lesquelles Ga, In et Tl.

Pour chaque type d'électrode on a calculé le degré d'intensification de la ligne, donné par le rapport entre le noircissement de la ligne pour les échantillons de pâte et le noircissement de la ligne pour les échantillons avec poudre

$$G = \frac{S_{\text{pâte}}}{S_{\text{poudre}}} \cdot \tag{1}$$

Les valeurs de G en fonction des types d'électrodes sont données dans le tableau 2.

Tableau 2

Nr.	Élément	Type d'électrode	Le noircissement $S \cdot 10^{-2}$		Degré d'intensification donné par (1)
			Poudre	Pâte	
1	Ga–4172 Å	1A	37,8	43,6	1,15
		1B	25,45	39,0	1,53
		2A	20,68	31,1	1,4
		2B	24,4	33,58	1,38
		3A	27,48	37,7	1,37
		3B	25,1	39,1	1,55
2	In–4511 Å	1A	64,54	70,1	1,08
		1B	43,94	72,9	1,65
		2A	46,5	57,4	1,23
		2B	48,82	63,32	1,28
		3A	50,26	58,8	1,16
		3B	49,4	65,1	1,31
3	Tl–5350 Å	1A	128,24	140,8	1,1
		1B	102,04	133,7	1,31
		2A	107,6	120,06	1,11
		2B	105,84	128,0	1,2
		3A	103,19	113,7	1,1
		3B	95,59	115,23	1,2

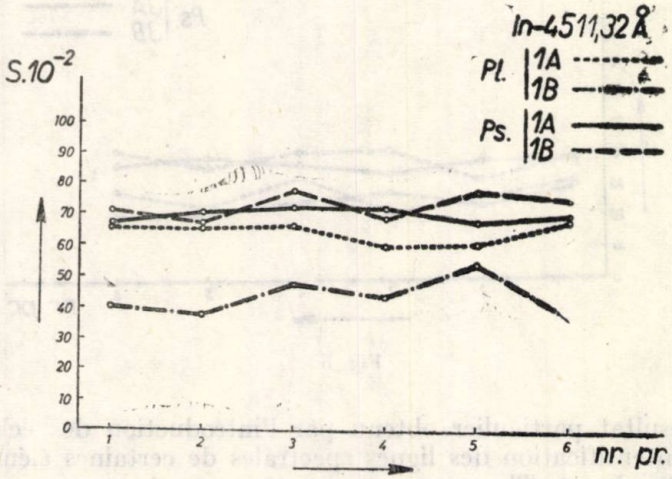

Fig. 6

Ainsi pour Ga—4 172 Å le degré d'intensification varie entre 1,15 avec l'électrode 1A et 1,55 avec l'électrode 3B, pour In—4 511 entre 1,08 avec 1A et 1,65 avec 1B, et pour Tl—5 350 entre 1,1 avec 1A et 1,31 avec 1B.

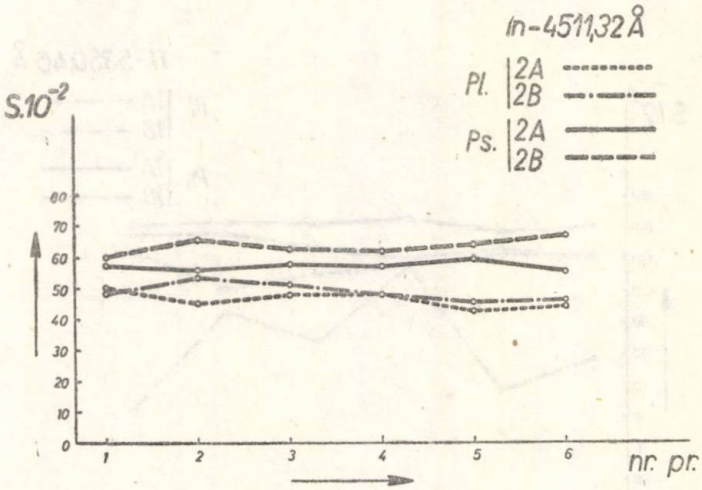

Fig. 7

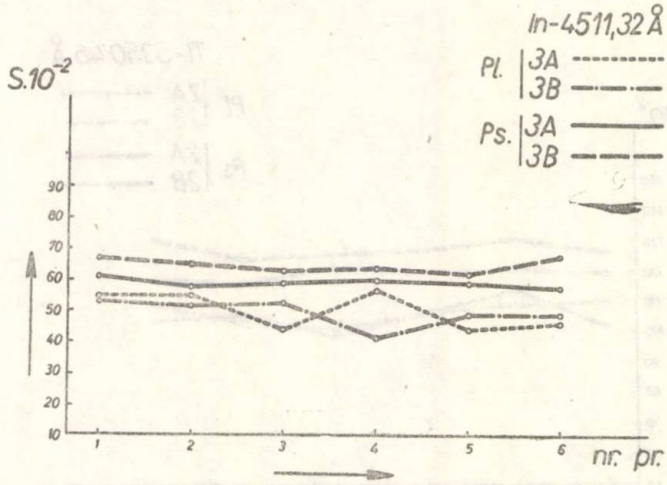

Fig. 8

Par l'introduction des échantillons sous forme de pâte dans l'analyse spectrographique de l'émission des traces de certains éléments des minéraux, la sensibilité de l'analyse augmente.

Parce qu'on observe aussi la dépendence entre les phénomènes de volatilisation des éléments dans le plasma de la décharge, en fonction de la forme des échantillons et du type de l'électrode, on a enregistré et mesuré les noircissements à différents moments, et on a tracé les courbes de volatilisation.

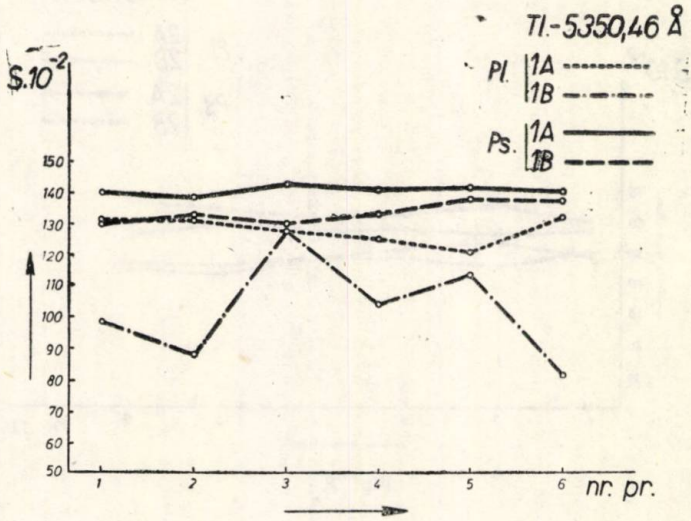

Fig. 9

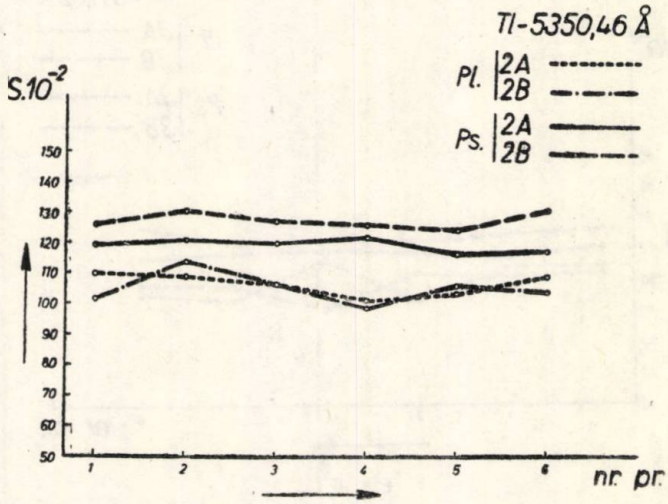

Fig. 10

Dans les fig. 12...14 sont données les courbes de volatilisation pour In—4511 Å.

Pour les types d'électrodes 1A, 1B, 2A, 2B, avec de la poudre, apparaîssent des maximas accentués du noircissement de la ligne après un intervale de temps de 8 s (4 s précombustion et 4 s combustion).

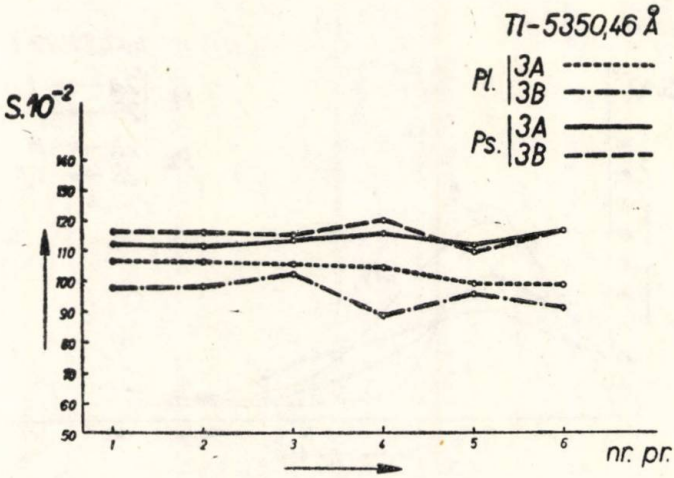

Fig. 11

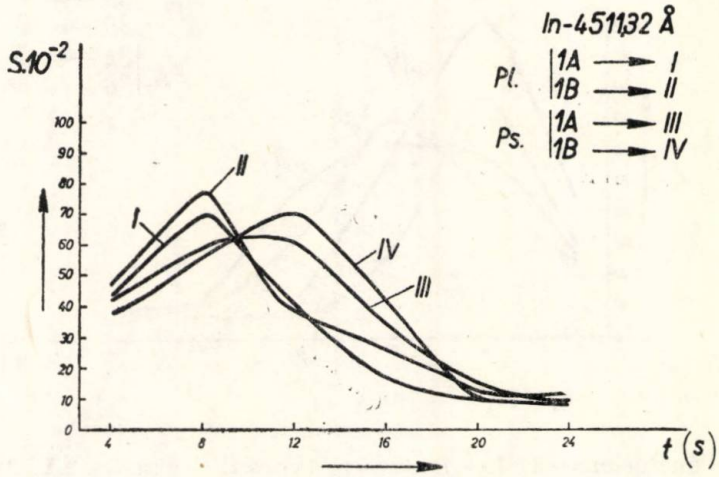

Fig. 12

Pour le type 3A avec de la poudre le maximum est enregistré après 12 s de combustion. Ces maximas indiquent l'existence d'une volatilisation rapide spécialement pour les types d'électrodes avec poudre 1A, 1B, 2B, 3A et 3B.

Dans le cas d'électrodes avec de la pâte, spécialement pour les types 2A et 3A on constate un applatissement des courbes, indiquant un diminution de la vitesse de volatilisation, donc une pénétration plus uniforme des éléments dans le plasma de la décharge.

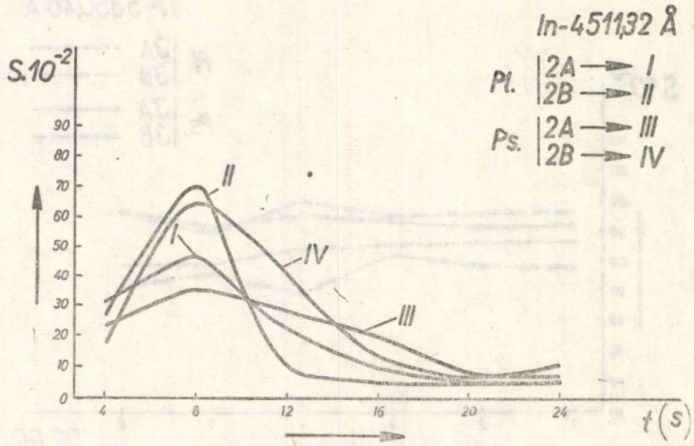

Fig. 13

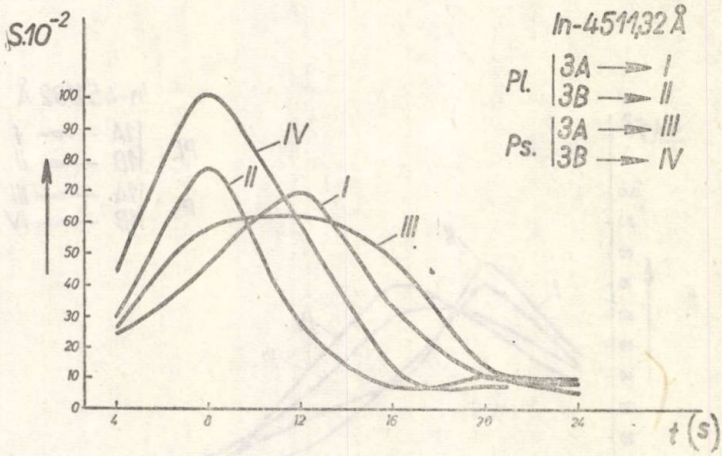

Fig. 14

5. Conclusions. 1. Les nouveaux types d'électrodes 2A, 2B, 3A, 3B, avec des échantillons sous forme de poudre réduissent les pertes d'échantillon à l'amorçage des décharges. Pour ces types d'électrodes on constate une légère amélioration de la reproduction des enregistrements.

2. L'introduction des échantillons sous forme de pâte, généralement pour tous les types d'électrodes, mène à une évidente amélioration de la reproduction des enregistrements et conduit à une augmentation de la précision des analyses quantitatives spectrographiques d'émission.

3. On constate que le plus grand noircissement des lignes a lieu dans tous les cas pour le type d'électrode 1A, en utilisant des échantillons sous forme de pâte.

4. Un résultat particulier est obtenu par l'introduction des échantillons sous forme de pâte, à savoir l'intensification de certains lignes spectrales qui conduit à une augmentation de la sensibilité de l'analyse et à une augmentation des limites de décélement de certains éléments.

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Institut Polytechnique, Jassy,
Chaire de Physique

BIBLIOGRAPHIE

1. Ahrens L. H., Taylor S. R., *Spectrochemical Analysis.* London, 1961.
2. Moenke H., *Spektralanalyse von Mineralien und Gesteinen.* Geest und Portig, Leipzig, 1962.
3. Ruthard S., *Chemische Spektralanalyse.* Springer Verlag, Berlin, 1970.
4. Русанов А. К., *Основы количественного спектрального анализа руд и минералов.* Изд. „Недра", М., 1971.
5. Slavin Morris, *Emission Spectrochemical Analysis.* Wiley-Interscience, New York—London — Sydney — Toronto, 1971,
6. Shoushiro Sakai, J. of the Spectroscopical Soc. of Japan, **15**, 55 (1966).
7. Moise M., Tacu Gh., Bul. Inst. polit. Iaşi, XV (XIX), *1 — 2*, 157 (1969).
8. Гусельников А. А., Русанов А. К., Ж. прикл. спектроск., **15**, *1*, 1—18 (1971).
9. Tacu Gh., Moise M., Bul. Inst. polit. Iaşi, XVIII (XXII), *1 — 2*, 156 (1972).

CERCETĂRI ASUPRA UZILIZĂRII UNOR TIPURI DE ELECTROZI ŞI A PROBELOR PASTĂ ÎN ANALIZA SPECTROGRAFICĂ DE EMISIE

(Rezumat)

Se prezintă o serie de rezultate obţinute prin introducerea unor noi tipuri de electrozi şi a utilizării probelor pastă, în cadrul analizei spectrografice a unor elemente (Ga, In, Tl) dintr-un minereu.

Noile tipuri de electrozi cu probe sub formă de pulbere micşorează pierderile de probă la amorsarea descărcării, ducînd la o anumită îmbunătăţire a reproductibilităţii înregistrărilor.

Introducerea probelor sub formă de pastă, o nouă metodologie în analiza spectrografică de emisie, produce o netă îmbunătăţire a reproductibilităţii înregistrărilor.

Un rezultat deosebit obţinut prin introducerea probelor sub formă de pastă îl constituie intensificarea unor linii spectrale, ceea ce măreşte limita de decelare a unor elemente

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INTERDEPENDENŢA ÎNTRE STRUCTURĂ ŞI CONTRACŢIA TERMICĂ A UNOR FIRE DE POLIETILENTEREFTALAT (PET)

DE

SOFIA MELINTE, M. LEANCĂ, MARCEL MOISE şi N. MATEESCU

1. Introducere. Proprietăţile firelor de PET depind de caracteristicile structurale ale macromoleculelor conţinute în polimer şi de condiţiile de prelucrare la formarea firelor. Intervenţia condiţiilor de prelucrare se manifestă prin importante modificări ale structurii polimerului.

S-a constatat că, de obicei, o proprietate nu este determinată de un singur factor structural şi totodată un anumit factor structural poate modifica un număr mai mare de proprietăţi. Din această cauză, datele obţinute cu privire la structura unui polimer, prin diferite metode de cercetare, sînt greu de corelat cu proprietăţile lui. Problema relaţiei între structură şi proprietăţi, prezintă o mare importanţă teoretică şi practică. Elaborarea unei teorii generale în această direcţie ar permite de exemplu, evitarea unor experimente empirice şi costisitoare pentru alegerea unor parametri tehnologici.

În această lucrare, s-a urmărit corelaţia între o serie de parametri structurali determinaţi de noi şi contracţia termică pentru firele de PET, obţinute în diferite condiţii.

2. Partea experimentală. În această lucrare, ca material de studiu s-a folosit firul filat de PET, care apoi a fost etirat sub diverse rapoarte de etirare la aproximativ 76°C.

2.1. *Determinarea birefringenţei.* Birefringenţa este o măsură a orientării moleculare totale a sistemului. Într-o probă orientată uniaxial, cum este cazul firelor de PET, birefringenţa este asociată cu diferenţa între indicii de refracţie ai celor două direcţii principale de vibraţie, a vectorului electric din unda polarizată liniar, respectiv paralel cu direcţia de etirare, $n_{\parallel}$ şi normal la această direcţie $n_{\perp}$,. Dacă toate moleculele anizotrope sînt distribuite la întîmplare în mostră adică mostra nu este orientată, media contribuţiei indicilor de refracţie va fi aceeaşi în toate direcţiile şi birefringenţa totală $\Delta n = n_{\parallel} - n_{\perp}$ va fi nulă. Dacă toate moleculele sînt perfect orientate paralel cu direcţia de etirare, birefringenţa va fi maximă. Pentru mostre orientate, birefringenţa va avea valori intermediare între aceste extreme, depinzînd de orientarea şi cantitatea fiecărui faze din mostră.

În cazul orientării uniaxiale (pentru fire considerate ca fiind formate din două faze : amorfă și cristalină) birefringența totală poate fi definită prin relația

$$\Delta_T = X\Delta_c^0 f_c + (1 - X)\,\Delta_{AM}^0 f_{AM} + \Delta_f\,, \tag{1}$$

unde Δ_c^0 este birefringența intrinsecă a regiunilor cristaline (a cristalitelor perfect orientate) și Δ_{AM}^0 — birefringența intrinsecă a regiunilor amorfe perfect orientate (a legăturilor amorfe), X este fracțiunea de material cristalin conținut în polimer care a fost determinată prin măsurători de densitate, Δ_f este birefringența de formă care exprimă devierile de la aditivitate și provine din deformarea cîmpului electric (din unda electromagnetică luminoasă) la frontiera între fazele cu formă anizotropică. Acestea pot fi de exemplu granițele între fazele cristalină și amorfă, sau pot fi granițele unor microgoluri în structura polimerului. În general contribuția acestei birefringențe este mică în comparație cu ceilalți doi termeni încît în această lucrare ea a fost neglijată nefiind determinabilă.

Orientarea cristalitelor în fir a fost determinată prin tehnica difracției de raze X și mărimea orientării lor este specificată prin funcțiile

$$f_c^{(2)} = \frac{3 < \cos^2\theta > -1}{2}\,, \tag{2}$$

unde θ este unghiul între axa c a cristalitei și direcția de etirare. Această funcție are valoarea zero pentru orientări întîmplătoare ale unității structurale (cristalitei, moleculei) și unu pentru o orientare perfectă (adică paralel cu direcția de tragere). Valorile obținute sînt date în tabelul 1.

Pentru determinarea birefringenței totale Δ_T s-au folosit două metode : metoda imersiei (liniei Becke) [10] și metoda compensării [10] folosind compensatorul de „calcit" tip Ehringhaus ce furnizează un domeniu de măsură de 133 ordine. Toate determinările s-au realizat la aceeași lungime de undă $\lambda = 546$ nm. Rezultatele sînt date în fig. 1 pentru metoda imersiei.

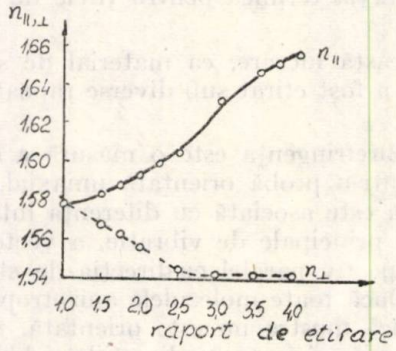

Fig. 1

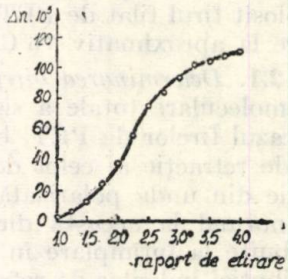

Fig. 2

Fig. 1 redă indicii de refracție pe cele două direcții pentru diverse rapoarte de etirare. Fiecare punct reprezintă o valoare medie a unui număr mare de determinări pentru același tip de probă. Diferența pe verticală între cele două curbe, estimează birefringența totală, redată în fig. 2.

Pentru determinarea orientării amorfe respectiv f_{AM} din relaţia (1) s-a utilizat $\Delta_c^0 = 0{,}220$ şi $\Delta_{AM}^0 = 0{,}275$ determinaţi de [1] şi X din determinări de densităţi.

2.2. *Determinarea densităţii.* Densitatea constituie unul din parametrii importanţi ce caracterizează starea de ordonare moleculară a unui polimer, implicit a unei fibre. Acest parametru este mult utilizat în estimarea cristalinităţii polimerilor filabili [5], [6] întrucît determinarea lui nu necesită aparatură complexă şi rezultatele sînt reproductibile. Conţinutul cristalin din polimer se poate calcula dacă se cunosc densităţile regiunilor cristaline ρ_c şi amorfă ρ_a precum şi densitatea medie a polimerului ρ. Din literatura de specialitate se cunoaşte pentru PET că $\rho_c = 1{,}455$ g/cm³ pentru 100% cristalin şi $\rho_a = 1{,}334$ g/cm³ pentru 100% amorf; determinîndu-se în cazul acestei lucrări densitatea probei prin metoda coloanei de gradient, s-a putut stabili proporţia în greutate a conţinutului cristalin, numită şi cristalinitate masică, prin relaţia

$$X_m = X = \frac{\rho - \rho_a}{\rho_c - \rho_a} \frac{\rho_c}{\rho}. \tag{3}$$

În fig. 3 este redată variaţia densităţii polimerului funcţie de raportul de etirare.

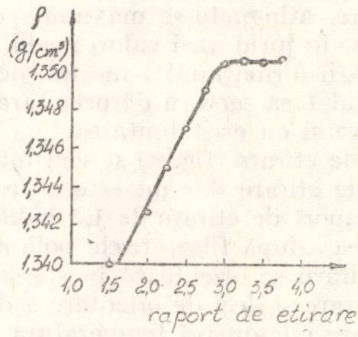

Fig. 3

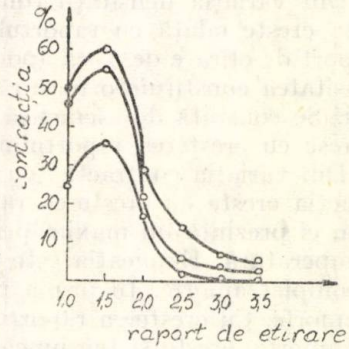

Fig. 4

2.3. *Influenţa contracţiei asupra orientării cristalitelor din polimer.* Contracţia firelor produsă prin intermediul diverselor fluide calde (aer, apă, ulei) se datoreşte relaxării forţelor intermoleculare, încît lanţurile moleculare mai mult sau mai puţin orientate ale polimerilor iau foarte probabil o anumită stare energetică, care duce la pierderea orientării şi răsucirii lor. Acest fapt duce de asemenea la reducerea lungimii firului. În produsul finit, acest fenomen este în general considerat de nedorit, deoarece duce la implicaţii negative în fabricarea produsului. Pentru a şti cum să acţionăm pentru înlăturarea efectelor nedorite asupra produselor finite este necesar să studiem comportarea firelor din care sînt confecţionate aceste produse, la diverse temperaturi.

S-a constatat că dacă firele de PET etirate sînt supuse unui tratament termic în aer cald sau apă, ele se contractă. Diferenţa între lungimea

iniţială şi finală a firului, ca urmare a efectului termic, este contracţia propriu-zisă. Determinarea acesteia s-a efectuat asupra firelor etirate sub diverse rapoarte de etirare, pentru diverse temperaturi (70°, 85°, 100°C) suspendate liber în apă timp de 15 minute.

S-a menţinut firul în apă un timp relativ scurt, insuficient ca apa să fie absorbită de probă, ceea ce modifică forţa de contracţie. Rezultatele obţinute sînt transpuse în graficele din fig. 4.

3. Discuţii. Din fig. 1 se constată că indicile de refracţie $n_{\parallel}$ creşte cu raportul de etirare, deci putem spune că acest parametru constituie o măsură a orientării formaţiunilor morfologice ale firelor. Indicele de refracţie transversal $n_{\perp}$ scade cu raportul de etirare, atingînd o valoare minimă pentru 2,5 ca apoi să se menţină aproximativ constant. Trebuie să reţinem faptul că, chiar în cazul unei orientări uniaxiale, apare totuşi şi o orientare de suprafaţă, şi că $n_{\perp}$ este de fapt media celor doi indici de refracţie principali, perpendiculari pe axa firului. Din fig. 2 se observă că birefringenţa totală creşte cu creşterea raportului de etirare pînă la un raport de 3,5, ca apoi creşterea să fie mult mai lentă. Aceasta arată că datorită creşterii etirării se orientează atît domeniile amorfe cît şi cele cristaline, şi pentru un anumit grad de etirare se produce un fel de saturaţie (adică cristalitele sînt complet orientate).

Din variaţia densităţii funcţie de raportul de etirare se observă că aceasta creşte odată cu raportul de etirare, atingîndu-se maximul pentru un raport de etirare de 3, ca apoi să oscileze în jurul unei valori medii. Deci şi densitatea constituie o măsură a orientării formaţiunilor morfologice ale firului. Se constată de asemenea din tabelul 1 că factorii de orientare f_c şi f_{AM} cresc cu creşterea raportului de etirare şi cu cristalinitatea.

Din variaţia contracţiei cu raportul de etirare (fig. 4) se constată că, contracţia creşte cu creşterea raportului de etirare dar nu este o creştere liniară, ci prezintă un maxim pentru un raport de etirare de 1,5 indiferent de temperatură. Explicaţia este următoarea : după filare firele poliesterice sînt complet amorfe. În prima parte a etirării se orientează în special zonele amorfe. Cu creşterea raportului de etirare şi deci de orientare a domeniilor amorfe, creşte şi tensiunea în fir, care micşorează temperatura cristalizării la rece (pentru PET amorf este 125°C) şi astfel firele tind către o structură cristalină. Pentru rapoarte de etirare mai mari, contractia este simţitor redusă, dar ea este totdeauna asociată cu dezorientarea considerabilă a regiunilor amorfe, ceea ce sugerează că aceasta constituie principalul mecanism al contracţiei.

Diverşi autori [7], [8] arată că pentru un raport de etirare de 2,5 —3, orientarea domeniilor amorfe se menţine aproape la acelaşi nivel, dar în acelaşi timp devine măsurabilă orientarea domeniilor cristaline, care manifestă o creştere lentă, concomitent cu creşterea gradului de etirare, fapt confirmat şi de determinările noastre (fig. 2 şi fig. 3), în care se observă că atît birefringenţa cît şi densitatea, cresc foarte lent după acest raport.

4. Concluzii. Etirarea firelor sintetice măreşte rezistenţa lor şi reduce elasticitatea. Între cristalinitate şi densitate există o strînsă corelaţie la firele etirate şi tratate termic. Cu ajutorul cristalinităţii şi a birefringenţei optice se poate calcula orientarea zonelor amorfe.

De remarcat este faptul că, contracţia este unul din parametrii importanţi care poate explica comportarea polimerilor înalţi etiraţi. Întrucît birefringenţa firelor contractate scade cu creşterea contracţiei, se poate conchide că modificarea dimensiunilor firului este datorată dezorientării moleculelor polimerului.

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Institutul politehnic, Iaşi,
Catedra de fizică

BIBLIOGRAFIE

1. Samuel R. J., J. Polymer Sci., part. A 2, **10**, 781 (1972).
2. Sweeting O. J., *The Science and Technology of Polymer Films*. Vol. 1.
3. Sack R. S., J. Polymer Sci., **54**, 543 (1961).
4. Desper C. R., Stein R. S., J. of Appleid Physics, **37**, *11*, 3990 (1966).
5. Ceauşescu E., Lebădă P., Hinte M., Materiale plastice, Elastomeri, Fibre sintetice, *12*, 615 – 622 (1970).
6. Fischer E. W. et al., Himicichsen Polymer, **7**, *4*, 195 – 198 (1966).
7. Новиков А. С., Иванова Р. С., Химические волокна, **6**, 54 (1961).
8. Dambeton J. H., J. Polymer Sci., A 2, **6**, 795 (1968).
9. Nobbs J. H. et al., J. Polymer Sci., **17**, *1*, 25 (1975).
10. Melinte S., şi colab., *Metode de determinare a birefringenţei optice la fibre*. Comunicare prezentată la Simpozionul C.F.S., Iaşi, 1976.

L'INTERDÉPENDANCE ENTRE LA STRUCTURE DES FIBRES DE PET ET LA CONTRACTION THERMIQUE

(Résumé)

On détermine divers paramètres structurels des fibres de PET comme la biréfringence, la densité, la cristalinité et les fonctions d'orientation en établissant une correlation entre ceux-ci et le mécanisme de l'étirage, la principale cause de l'orientation du polymère. On détermine aussi la contraction thermique des fibres de PET, paramètre important qui modifie la structure interne du polymère.

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